Computer algebra experimentation with Higmanian rank 5 association schemes on 40 vertices and related combinatorial objects

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Dedicated to the memory of Donald G. Higman.

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1 Introduction

Paying our tribute to the mathematical heritage of D. G. Higman we investigate imprimitive association schemes on 40 points with 4 classes which belong to proper class II in a sense of [Hig95].

Considering four possible sets of intersection numbers, corresponding to a parabolic of type $10 \circ K_4$, we enumerate all 15 schemes for the first case, discover schemes covering second and third parameter sets, proving that second case is unique up to isomorphism, while the fourth parameter set is easily discarded.

Though most of the obtained results are essentially computer dependent, finally for many constructed structures a computer free interpretation is presented.

Section 2 provides a short introduction to the concepts of coherent coniguration and association scheme.

Following to a spirit of H. S. M. Coxeter, a number of nice auxiliary objects (including cages on 40 and 50 vertices) are inspected in Section 3, enjoying all opportunities provided by the use of modern computer algebra techniques.

In Section 4 a new concept of a total graph coherent configuration suggested by the authors is investigated. Exceptional mergings which appear from such configuration on 40 points serve as a motivation for the whole project. This in particular implies interest to Higman's classification of rank 5 imprimitive association schemes which is briefly discussed in Section 5 and to the consideration of the classical Deza graph (a generalization of the notion of a strongly regular graph) in Section 6. A suggested new model for this object and a number of related structures stem from investigation of suitable permutation actions of group $E_{2^4} \rtimes S_5$ of order 1920.

The coherent closure of the classical Deza graph on 40 vertices provides an attractive example of a Higmanian association scheme $\mathfrak{m}$. There are altogether 15 schemes which are algebraically isomorphic to $\mathfrak{m}$: an innovative computer aided approach to their enumeration is described in Section 7. Elements of recently developed theory of WFDF coherent configurations are considered in Sections 8, 9, in particular, the scheme $\mathfrak{m}$ and two of its algebraic twins are revealed as merging schemes inside of certain WFDF configuration.

Section 10 justifies newly discovered properties of an amazing graph on 40 vertices: Anstee-Robertson cage of valency 6 and girth 5 which was also discovered independently by C. W. Evans. The coherent closure of this graph is a non-Schurian association scheme of rank 5. This unexpected structure turns out to be unique up to isomorphism: a nice simple consequence of the uniqueness of the cage on 40 vertices.

An interesting link of this cage with the unique locally icosahedral graph on 40 vertices (see [BloBBC85]) is described.

The cage on 40 vertices was originally discovered by Robertson as an induced subgraph of the Hoffman-Singleton graph, HoSi. A few models of HoSi are revised in Section 11 in terms of coherent configurations. The considered structures may be of a certain independent interest in view of ongoing attempts to find any similarities between HoSi and possible Moore graph of valency 57. In addition, two more association schemes on 40 vertices are briefly introduced.

Finally, Section 12 contains discussion of various issues which were postponed to the end from the main line of presentation based on amalgamation of techniques in algebraic combinatorics, group theory and computer algebra.

2 Coherent configurations and association schemes

In this section we provide brief discussion of most significant notions and notation in order to make presentation relatively self-contained. A survey [FarKM94] may serve as a source for more details. We denote dihedral group of order $2n$ by D_n. A cyclic group of order n is denoted by $\mathbb{Z}_n$ in contrast to C_n which is a regular connected graph of valency 2 with n vertices.

2.1 Main concepts

We start with a brief review of main concepts referring to [BanI84], [BroCN89], [FarKM94] for a more detailed presentation.

By a *color graph* we understand a pair $(\Omega, \mathcal{R})$, where Ω is a set of vertices and $\mathcal{R}$ a partition of Ω^2 into a set of non-empty disjoint binary relations on Ω.

According to Higman [Hig70] a *coherent configuration* is a color graph $\mathfrak{m} = (\Omega, \mathcal{R})$, $\mathcal{R} = \{R_i | i \in I\}$ such that the following conditions are satisfied:

1. The identity relation $Id_\Omega = \{(x,x) | x \in \Omega\}$ is a union of suitable relations R_i, $i \in I'$, $I' \subseteq I$.

2. For each $i \in I$ there exists $i' \in I$ such that $R_i^t = R_{i'}$, where $R_i^t = \{(y,x) | (x,y) \in R_i\}$.

3. For any $i, j, k \in I$ the number p_{ij}^k of elements $z \in \Omega$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$ is constant provided that $(x,y) \in R_k$.

The numbers p_{ij}^k are called *intersection numbers* of $\mathfrak{m}$. We refer to R_i as *basic relations*, graphs $\Gamma_i = (\Omega, R_i)$ as *basic graphs* and adjacency matrices $A_i = A(\Gamma_i)$ as *basic matrices* of $\mathfrak{m}$.

We use notions of a fiber of a coherent configuration, and of a type of it as they appear, e.g. in [Hig75]. The number $|I|$, equal to the number of basic relations is called *rank* of $\mathfrak{m}$, the number $|\Omega|$ is the *order* of $\mathfrak{m}$.

If (G, Ω) is a permutation group, $2 - orb(G, \Omega)$ denotes the set of all *2-orbits* of (G, Ω), that is orbits of the induced action of G on Ω^2. It is easy to see that $(\Omega, 2 - orb(G, \Omega))$ is a coherent configuration. Such coherent configurations will be called *Schurian* (cf. [FarKM94]).

A particular case of a coherent configuration $\mathfrak{m}$ for which the identity relation Id_Ω is one of the basic relations of Ω is called a *homogeneous coherent configuration* or an association scheme. Typically a basic relation Id_Ω is denoted by R_0, all the remaining basic relations are called *classes*. We stress that like in [BanI84] in our terminology an association scheme is not presumed to be symmetric or commutative.

A notion of a distance regular graph and of a corresponding to it metric (P-polynomial) association scheme is considered in [BanI84], [BroCN89].

Strongly regular graphs are distance regular graphs of diameter 2.

2.2 Coherent closure

A notion of a coherent configuration may be reformulated in terms of matrices. A *coherent algebra* W is a set of square matrices of order n over the field $\mathbb{C}$ which is a matrix algebra and in addition is closed with respect to Schur-Hadamard multiplication, transposition and contains the identity matrix I and the all-one matrix J.

The set of basic matrices $\{A_i|i \in I\}$ of a coherent configuration $\mathfrak{m}$ serves as a *standard basis* of the corresponding coherent algebra W, in this case we write $W = \langle A_0, A_1, \ldots, A_{r-1}\rangle$. Conversely, a coherent algebra W has a basis of (0,1)-matrices, and this basis can be regarded as a set of basic matrices of a coherent configuration $\mathfrak{m}$.

It is easy to check that the intersection of coherent algebras is again a coherent algebra. This implies existence of the smallest coherent algebra containing a prescribed set B of matrices of order n. Such algebra is called *coherent closure* of B, denoted by $\langle\langle B\rangle\rangle$. An efficient polynomial time algorithm for the computation of coherent closure was suggested in [WeiL68], see also [Wei76]. Usually we call this algorithm WL-stabilization (see e.g. [KliRRT99]).

A graph $\Gamma = (V, E)$ will be called by us a *coherent graph* if E is one of the the basic relations of the coherent closure $\langle\langle\Gamma\rangle\rangle$. For example, each distance regular graph is coherent.

2.3 Isomorphisms and automorphisms

An *isomorphism* of color graphs $(\Omega, \mathcal{R})$ and $(\Omega', \mathcal{R}')$ is a bijection ϕ from Ω to Ω' which induces a bijection of colors (relations) in $\mathcal{R}$ onto colors in $\mathcal{R}'$. A *weak* (or *color*) *automorphism* of $\Gamma = (\Omega, \mathcal{R})$ is isomorphism of Γ with itself. If the induced permutation of colors is the identity permutation then we speak of a *(strong) automorphism*.

We denote by $CAut(\Gamma)$ and $Aut(\Gamma)$ the groups of all weak and strong automorphisms of Γ, respectively. Clearly, $Aut(\Gamma) \trianglelefteq CAut(\Gamma)$. In case when Γ is a Schurian coherent configuration, the group $CAut(\Gamma)$ coincides with the normalizer of $Aut(\Gamma)$ in symmetric group $S(\Omega)$.

An *algebraic isomorphism* between coherent configurations $(\Omega, \mathcal{R})$ and $(\Omega', \mathcal{R}')$ is a bijection $\phi : \mathcal{R} \to \mathcal{R}'$ such that for all $i, j, k \in I$, $p_{ij}^k = p_{i^\phi j^\phi}^{k^\phi}$. An algebraic isomorphism of a coherent configuration $\mathfrak{m} = (\Omega, \mathcal{R})$ with itself is called an *algebraic automorphism* of $\mathfrak{m}$. The group of algebraic automorphisms of $\mathfrak{m}$ is denoted by $AAut(\mathfrak{m})$.

Clearly, $CAut(\mathfrak{m})/Aut(\mathfrak{m}) \leq AAut(\mathfrak{m})$. If the quotient group $CAut(\mathfrak{m})/Aut(\mathfrak{m})$ is a proper subgroup of $AAut(\mathfrak{m})$ then the algebraic automorphisms of $\mathfrak{m}$ which are not induced by $\phi \in CAut(\mathfrak{m})$ are called *proper* algebraic automorphisms. We refer to [KliMPWZ07] for a detailed consideration of this concept.

2.4 Mergings

If W' is a coherent subalgebra of a coherent algebra W, then the corresponding coherent configuration $\mathfrak{m}'$ is called a *fusion* (or *merging configuraition*) of $\mathfrak{m}$. (Note that in many cases we abuse notation refering to W and m as to the same object).

In case when $\mathfrak{m} = (\Omega, 2 - orb(G, \Omega))$ for a suitable permutation group G, overgroups of G in $S(\Omega)$ provide a natural origin for fusions of $\mathfrak{m}$. Thus, most interesting in a sense fusions are *non-Schurian* ones, that is those which do not emerge from a suitable overgroup of (G, Ω).

For each subgroup $K \leq AAut(\mathfrak{m})$, its orbits on the set of relations define a merging coherent configuration, which is called *algebraic merging* defined by K. Again, those algebraic mergings which are non-Schurian are of a special interest as less predictable combinatorial objects.

If W' and W'' are coherent subalgebras of a coherent algebra W, such that W' and W'' are not isomorphic and in addition there exists $\phi \in AAut(W)$ that maps W' to W'', then W' and W'' form a pair of *twins* inside of W.

2.5 Decomposable schemes

Though operations of tensor (direct) and wreath product may be defined for arbitrary coherent configurations, we will restrict our consideration only to the case of association schemes. They may be regarded as combinatorial analogues of similar operations over permutation groups.

If W_1, W_2 are homogeneous coherent algebras of orders n_1, n_2 and ranks r_1, r_2 respectively, then their tensor product $W_1 \otimes W_2$ is homogeneous coherent algebra of order $n_1 n_2$ and rank $r_1 r_2$.

Let $\mathfrak{m}_1 = (\Omega_1, R)$ and $\mathfrak{m}_2 = (\Omega_2, S)$ be two association schemes of orders n_1, n_2 and ranks r_1, r_2, respectively. Let $\Omega = \Omega_1 \times \Omega_2$. We define on Ω basic relations Id_Ω, $\Delta_i = \{((u_1, v_1), (u_2, v_2)) | (u_1, u_2) \in R_i, 1 \leq i \leq r_1 - 1\}$ and $\Theta_1 = \{((u, v_1), (u, v_2)) | (v_1, v_2) \in S_j, u \in \Omega_1, 1 \leq j \leq r_2 - 1\}$. The systen $\mathfrak{m} = (\Omega, \{Id_\Omega\} \cup \{\Delta_i | 1 \leq i \leq r_1 - 1\} \cup \{\Theta_j | 1 \leq j \leq r_2 - 1\})$ turns out to be an association scheme of order $n_1 n_2$ and rank $r_1 + r_2 - 1$. We use notation $\mathfrak{m} = \mathfrak{m}_1 wr \mathfrak{m}_2$ (or $\mathfrak{m} = \mathfrak{m}_1 \wr \mathfrak{m}_2$) and call m the *wreath product* of $\mathfrak{m}_1$ and $\mathfrak{m}_2$.

Note that a more general definition of wreath product, see e.g. [Wei76], [EvdPT00] provides an association scheme which is algebraically (but not necessarily combinatorially) isomorphic to $\mathfrak{m}_1 \wr \mathfrak{m}_2$.

An association scheme is called *primitive* if all its non-reflexive basic graphs are connected, otherwise it is *imprimitive.* An association scheme is imprimitive if and only if it admits non trivial equivalence relations as a union of suitable basic relations. Such equivalence relations are called *imprimitivity systems* ([BanI84]), alternative names are *parabolics* ([Hig95]), *closed sets* ([Zie96]). For each imprimitive association scheme and for each imprimitivity system σ we may define a *quotient scheme* $\mathfrak{m}/\sigma$ on the sets of classes of equivalence relation σ. Following Higman, for a pair $(\mathfrak{m}, \sigma)$ we speak of its *rank* as a rank of association scheme induced by $\mathfrak{m}$ on an arbitrary class of σ. In this case, *corank* is rank of $\mathfrak{m}/\sigma$. The sum of the rank and corank of $(\mathfrak{m}, \sigma)$ is at most $r + 1$ where r is the rank of $\mathfrak{m}$, with equality if and only if $\mathfrak{m}$ is algebraically isomorphic to $\mathfrak{m}_1 \wr \mathfrak{m}_2$, where $\mathfrak{m}_1$ is the quotient scheme, while m_2 is isomorphic to an induced scheme on classes of σ.

An imprimitive association scheme is called *decomposable* if it can be represented as tensor or (generalized) wreath product of smaller schemes. Decomposable schemes (together with imprimitive rank 3 schemes) are commonly regarded as trivial objects, real interest in theory of association schemes starts from the investigation of the indecomposable objects (including primitive schemes of rank at least 3).

2.6 Equitable partitions

Let $\Gamma = (V, E)$ be a graph. A partition $V = \{V_1, \ldots, V_s\}$ of V is called *equitable* with respect to Γ if for all $k, l \in \{1, \ldots, s\}$, the numbers $|\Gamma(v) \cap V_l|$ are constant for all $v \in V_k$. Here, $\Gamma(v) = \{u \in V | \{u, v\} \in E\}$ is the neighbor set of vertex v. Usually an equitable partition of a graph is accompanied by *intersection diagram*, which is a kind of quotient graph on which all numbers $|\Gamma(v) \cap V_l|$ are depicted.

A typical source of equitable partitions is orbits on V of a subgroup of the automorphism group of a graph.

The following proposition should in our eyes be regarded as a folklore one.

Lemma 2.1. *Let $\mathfrak{m} = (\Omega, \{R_i | i \in I\})$ be an association scheme and $\tau = \{\tau_0 = \{0\}, \tau_1, \ldots, \tau_s\}$ be a partition of I. Define $S_j = \bigcup_{i \in \tau_j} R_i$ for all $0 \leq j \leq s$. Let $\Gamma_j = (\Omega, S_j)$, $0 \leq j \leq s$. Let $x \in \Omega$ be a reference vertex and $\sigma = \{\{x\}, \Gamma_1(x), \ldots, \Gamma_s(x)\}$ be a partition of Ω into the neighbor sets of x in the graphs $\Gamma_0, \ldots, \Gamma_s$.*

Assume that relations $S_1, \ldots, S_s$ are symmetric and the adjacency matrices $A(\Gamma_i)$ for $2 \leq i \leq s$ are expressible as suitable polynomials in $A(\Gamma_1)$. If σ is an equitable partition with respect to Γ_1 then $\mathfrak{m}' = (\Omega, \{S_j | j \in [0, s]\})$ is a merging association scheme of $\mathfrak{m}$. Moreover, in this case $\langle\langle \Gamma_1 \rangle\rangle = \mathfrak{m}'$.

2.7 Computer tools

The following computer programs were used by us in order to find new combinatorial objects, and to investigate algebraic properties (such as properties of the automorphism group) of known and new objects.

2.7.1 COCO

COCO is a set of programs for dealing with coherent configurations.

Developed in 1990-2, Moscow, USSR, mainly by Faradžev and Klin [FarK91], [FarKM94].

The programs include:

ind - a program for calculating induced action of a permutation group on a combinatorial structure;

cgr - a program to calculate the centralizer algebra of a permutation group;

inm - a program to calculate the intersection numbers (also known as structure constants) of a coherent configuration;

sub - a program to find fusion association schemes of a coherent configuration given by its structure constants;

aut - a program to calculate automorphism groups of a coherent configuration and of its fusion association schemes.

Usually, those programs are used in the above order. This provides a computerized way to find all association schemes invariant under a given permutation group, together with their automorphism groups.

2.7.2 WL-stabilization

Two implementations of the Weisfeiler-Leman stabilization [Wei76] are available. WL-stabilization is an efficient algorithm for calculating coherent closure of a given matrix [BabCKP97].

Using available programs, in many cases we have a limit for the possible order or rank of the coherent closure, or are only interested in finding out some lower bound for the rank of the closure, in which case an ad hoc simple calculation is sufficient.

2.7.3 GAP

GAP [GAP], [Sch95], an acronym for "Groups, Algorithms and Programming", is a system for computation in discrete abstract algebra. The system supports easy addition of extensions (packages, in gap nomenclature), that are written in the GAP programming language, which can add new features to the GAP system.

One such package, which is very useful in algebraic combinatorics is GRAPE [Soi93]. It is designed for construction and analysis of finite graphs. GRAPE itself is dependent on an external program, nauty [McK90] in order to calculate the automorphism group of a graph.

Another package is DESIGN, used for construction and examination of block designs.

GAP is used in the course of investigations in algebraic combinatorics in order to:

Construct incidence structures (graphs, block designs, geometries, coherent configurations, etc).

Calculate automorphism groups of such structures.

Check regularity properties and parameters of structures.

Find cliques in graphs, and substructures of given structures in general.

Find abstract structure of a group as well as identification of a permutation group.
Find conjugacy classes of elements and of subgroups of a group.

2.7.4 COCO v.2

While a lot of calculation in algebraic combinatorics are done in GAP, some algorithms or operations are only available in certain other programs discussed above. This results in permanent necessity to translate the output of one program to a format that is acceptable as input of the other program.

The COCO v.2 initiative aims to re-implement the algorithms in COCO, WL-stabilization and DISCRETA as a GAP package. In addition, the new package should essentially extend abilities of current version, basing on new theoretical results obtained since the original COCO package was written.

3 Preliminaries

3.1 Möbius-Kantor configuration 8_3

A configuration 8_3 is a regular uniform incidence structure with the parameters $(v, b, k, r) = (8, 8, 3, 3)$ which forms a partial linear space, that is any two distinct points are incident to at most one line. A classical model of 8_3 is formed by non-zero vectors of the vector space of dimension 2 over $\mathbb{G}F(3)$, considered as points while lines are sets of the form $U + v$, where U is any subspace of dimension one of $\mathbb{G}F(3)^2$, and v is any non-zero vector.

It immediately follows from the given construction that the point graph of 8_3 admits group $\mathrm{GL}(2, 3)$ of order 48 as subgroup of its automorphism group. Moreover, $\mathrm{GL}(2, 3)$ is the whole automorphism group.

In fact, 8_3 is unique up to isomorphism, therefore it is self dual. This implies that the automorphism group of the Levi (incidence) graph of 8_3 is group H of order 96 which is isomorphic to $\mathrm{GL}(2, 3) : 2$. We will call this Levi graph the Möbius-Kantor graph, or briefly MK-graph. The description and characterization of 8_3 goes back to XIXth century. We refer to [Cox77], [CosL01] for more information about this structure.

A diagram of MK-graph in a form of generalized Petersen graph, briefly gP-representation (in sense of [FruGW71]) is depicted in Figure 3.1(a).

Another diagram provides a Hamiltonian cycle in the same graph. Both diagrams (up to labeling) are borrowed from [Cox77]. The reader is invited to check that we indeed work with a copy of MK-graph. For this purpose, using black and white labels, we reveal the bipartite partition of the depicted graph and recognize in it a Levi graph of a suitable partial linear space (note that the graph does not contain quadrangles.)

The defined group H of order 96 acts transitively on 16 vertices of MK-graph. It has 3 subgroups of index 2. One of this groups, which stabilizes one

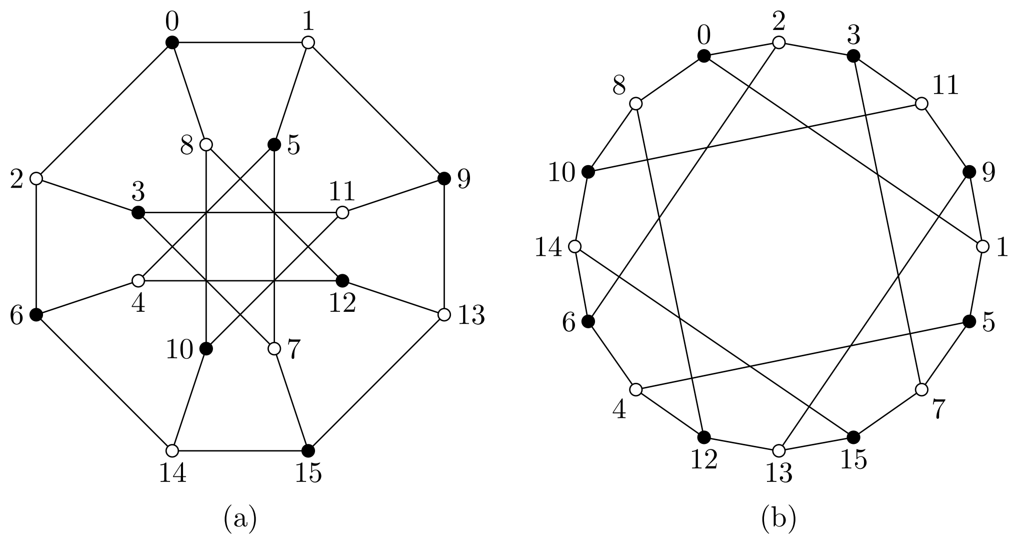

Figure 3.1: Two diagrams of MK-graph.

part of the bipartite partition is intransitive group isomorphic to $\mathrm{GL}(2,3)$.

Note that $\mathrm{GL}(2,3)$ has unique (up to conjugacy) transitive action of degree 16 on the cosets of a Sylow subgroup of order 3. Let us denote it by K. This permutation group K can be regarded as a subgroup of H. To justify this claim, consider action of $Aut(8_3)$ on cyclically ordered lines of 8_3 and define on these objects (regarded as vertices) the same structure of Levi graph of 8_3.

In [Cox77], Coxeter suggested a few ways to recognize inside of this group K the stabilizer, L, of gP-representation which has an easy geometrical explanation. We will follow these ways, using a moderate modification of Coxeter's arguments.

Clearly, gP-representation may be regarded as union of two octagons (which are induced subgraphs of MK-graph) and a 1-factor. Let us call these structures gP-octagons and gP-1-factor respectively. By stabilizer of any of these structures we mean the subgroup of H which preserves the structure as a whole.

Proposition 3.1. *a) Stabilizer L of gP-representation is a group of order 32 which acts transitively on the vertices.*

b) Group L is also the stabilizer of gP-1-factor.

c) There are exactly three gP-representations of the MK-graph. The orbit of gP-1-factor (under the action of H) has length 3.

d) The three gP-1-factors forming the orbit under H are the 1-factors compatible with (an arbitrary) gP-representation.

e) Six gP-octagons form one orbit under H.

f) Stabilizer M of a gP-octagon is dihedral group D_8 of order 16 acting faithfully on the vertices of the octagon.

g) 12 directed gP-octagons which appear as orientation of a gP-octagon are split into two orbits of length 6 under the action of K.

h) K may be interpreted as the stabilizer in H of one orbit of directed gP-octagons.

i) K contains a regular subgroup which is isomorphic to the quasidihedral (or semidihedral) group of order 16, QD_{16}.

j) MK-graph can be represented as a Cayley graph over QD_{16}.

Proof. We provide a few details of the proof. The remaining arguments are straight forward (in the proposed sequence of claims).

a) Let $g_1 = (0, 1, 9, 13, 15, 14, 6, 2)(8, 5, 11, 12, 7, 10, 4, 3, 8)$,
$g_2 = (0, 1)(2, 9)(6, 13)(14, 15)(5, 8)(3, 11)(4, 12)(7, 10)$,
$g_3 = (0, 3, 15, 12)(1, 11, 14, 4)(2, 7, 13, 8)(5, 9, 10, 6)$. Simple visual inspection confirms that $g_1, g_2, g_3 \in L$. It is also clear that $|\langle g_1, g_2, g_2\rangle| = 32$. On the other hand it is evident (from geometrical arguments) that $|L| \leq 32$. Thus, $L =< g_1, g_2, g_3 >$ has order 32.

d) Split two gP-octagons on Figure 3.1(a) into two 1-factors. Get together with corresponding gP-1-factor a set consisting of three 1-factors. This is a desired orbit of gP-1-factors.

f) $M =< g_1, g_2 >$.

g) Consider group $N =< g_1 >$, this is the full stabilizer of directed gP-octagon $(0, 1, 9, 13, 15, 14, 6, 2)$ in group H. As a subgroup of K, N has index 6.

h) Consider 6 octagons $(0, 1, 9, 13, 15, 14, 6, 2)$, $(0, 8, 10, 14, 15, 7, 5, 1)$, $(0, 2, 3, 7, 15, 13, 12, 8)$, $(1, 5, 4, 6, 14, 10, 11, 9)$, $(3, 11, 10, 8, 12, 4, 5, 7)$, $(2, 6, 4, 12, 13, 9, 11, 3)$.

   Consider g_1 and $g_4 = (1, 8, 2)(3, 9, 10)(5, 12, 6)(7, 13, 14)$. A routine inspection reveals that g_1, g_4 preserve the system of above six directed gP-octagons. Moreover, group $< g_1, g_4 >$ acts transitively on this system. Thus, according to (g) the full stabilizer is a transitive subgroup of H of index 2. Group H has two such subgroups. The one different from $\mathrm{GL}(2, 3)$ does not contain elements of order 8. In fact, $K =< g_1, g_4 >$.

i) Let $g_5 = (0, 4)(1, 5)(2, 12)(3, 13)(6, 8)(7, 9)(10, 14)(11, 15)$,
$g_6 = (0, 1, 9, 13, 15, 14, 6, 2)(3, 8, 5, 11, 12, 7, 10, 4)$ and consider group

$< g_5, g_6 >$. Simple inspection with the aid of Figure 3.1 (a) shows that this group stabilizes depicted gP-representation, has index 2 in L, is transitive, and $g_5 g_6 g_5 = g_6^3$, therefore it is isomorphic to QD_{16}, the quasidihedral group of order 16.

□

Remarks.

1. Clearly, the suggested proof is Deus ex machine. An alternative may be to use generators and defining relations as was done originally by Coxeter.

2. In general (computations with the aid of GAP) MK-graph has three orbits of 1-factors of lengths 24,6,3, and two orbits of induced octagons of lengths 24,6. The objects exploited by us are most symmetric and thus easily visible.

3. MK-graph may be represented as a regular map of type $\{8, 3\}$ on an orientable surface of genus 2. The orbit of 6 oriented gP-octagons used by us is nothing else but those (up to labeling) 6 octagons which present a map depicted in Figure 20 in [Cox77].

4. The group K is the subgroup of index 2 in H which preserves orientation of the map on the surface.

3.2 4-dimensional cube

We consider well-known structure of the 4-dimensional cube Q_4. The vertices of Q_4 are binary sequences of length 4, two sequences are adjacent if they differ in exactly one position. It is convenient to use also a canonical in a sense labeling of the vertices of Q_4 by numbers from $[0, 15]$, where each number is decimal representation of corresponding binary sequence.

Q_4 is bipartite graph with girth 4. All quadrangles of Q_4 have an evident geometrical sense: two from four binary coordinates take a prescribed value while the remaining two coordinates vary. Clearly, Q_4 has $\binom{4}{2} \cdot 2^2 = 24$ quadrangles.

It immediately follows from the definition that Q_4 may be represented as a Cayley graph over group E_{2^4} with a connection set
$X_4 = \{0001, 0010, 0100, 1000\}$. The full automorphism group P of Q_4 is the exponentiation $S_2 \uparrow S_4$ of symmetric group S_2 with S_4 of order $2^4 \cdot 4! = 384$, cf. [KliPR88].

The stabilizer of a quadrangle in group P is easily identified with a group $D_4 \times \mathbb{Z}_2$ of order 16 with the orbits on vertices of length 4, 4 and 8.

According to GAP, Q_4 has 272 1-factors in 8 orbits of length 4, 48, 24, 48, 96, 12, 32, 8. We are interested in the two small orbits.

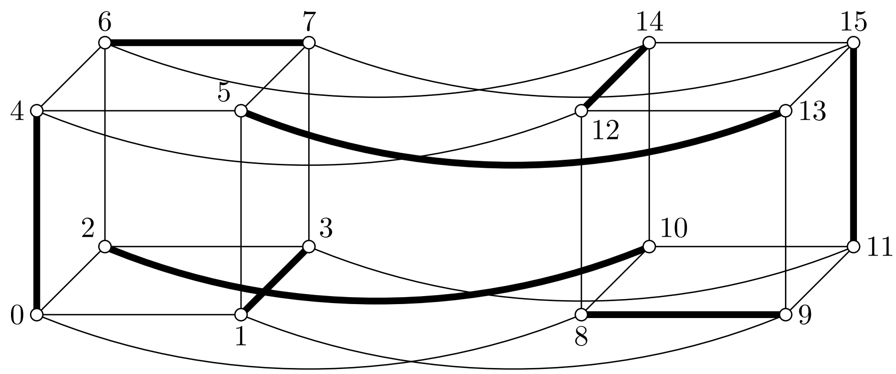

Figure 3.2: 4-dimensional cube Q_4 with a skew 1-factor

The orbit of length 4 is easily recognized as consisting of Cayley graphs over E_{2^4} with a connection set $\{x\}$, $x \in X_4$. We will call these spanning subgraphs of Q_4 *direct 1-factors*. Removal of direct 1-factor evidently splits Q_4 into two disjoint copies of 3-dimensional cube Q_3.

The orbit of length 8 will be called the orbit of *skew 1-factors*, a representative of it is visible in Figure 3.2 (bold edges).

An explanation of structure of a skew 1-factor: Remove a direct 1-factor, and start from resulting pair of disjoint copies of Q_3. Take a pair of antipodal vertices in one copy of Q_3 (say, $\{2, 5\}$). The remaining 6 vertices of the same Q_3 form an induced hexagon (automorphic subgraph in sense of [JonKL00]). Split the hexagon into two copies of $3 \circ K_2$, take one of them (with edge set $\{\{0, 4\}, \{1, 3\}, \{6, 7\}\}$). The neighbors of remaining $3 \circ K_2$ in corresponding direct 1-factor give three more edges ($\{8, 9\}$, $\{11, 15\}$, $\{12, 14\}$). Now there is unique way to add two more edges in order to get the depicted skew 1-factor. Easy combinatorial counting shows that there are $\frac{4 \cdot 2 \cdot 2}{2} = 8$ different possibilities to get skew 1-factor. All skew 1-factors are isomorphic with respect to the group $P = Aut(Q_4)$. Therefore the stabilizer K' of a skew 1-factor is a subgroup of order 48 of the group P.

To identify this group K' we suggest the reader to remove from Q_4 the skew 1-factor depicted in Figure 3.2 and to notice that the remaining subgraph exactly coincides with a copy of MK-graph in Figure 3.1. Therefore we conclude that K' is a subgroup of index 2 in H.

To distinguish which of the three subgroups we face, let us now add to the copy of MK-graph in Figure 3.1(a) the same skew 1-factor. We get the graph in Figure 3.3.

We stress the following features of the depicted figure:

- The evident horizontal symmetry of the MK-graph transforms the de-

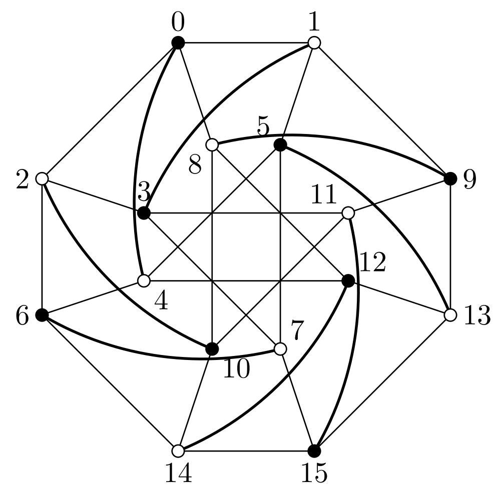

Figure 3.3: MK-graph together with a skew 1-factor of Q_4

picted skew 1-factor of Q_4 to another one, namely

$$\{\{0,11\},\{1,12\},\{2,5\},\{3,14\},\{4,15\},\{6,8\},\{7,9\},\{10,13\}\}$$

in another copy of Q_4.

- The colored graph in Figure 3.3 has a transitive automorphism group (check that the regular group QD_{16} presented in Section 3.1 preserves the graph).
- A permutation g_1 generating cyclic subgroup of order 8 belongs to K'.

All this information immediately implies that the full automorphism group K' of the color graph depicted in Figure 3.3 coincides with the group $K = \mathrm{GL}(3,2)$.

Proposition 3.2. *a) The group $P = Aut(Q_4)$ contains a transitive subgroup K which is isomorphic to* $\mathrm{GL}(3,2)$.

b) $K = \mathrm{GL}(3,2)$ may be characterized inside of P as the stabilizer of a skew 1-factor.

c) Q_4 contains exactly 8 skew 1-factors forming an orbit under action of P.

Proof. As an alternative to the arguments presented above, we suggest the reader to consider again regular group $QD_{16} = \langle g_5, g_6\rangle$ and to detect its four 2-orbits with the representatives $(0,1)$, $(0,2)$, $(0,4)$, $(0,8)$. Union of these 2-orbits provides the arc set of Q_4. While first and second orbits consist of two directed gP-octagons, the fourth is the gP-1-factor, and the third is the

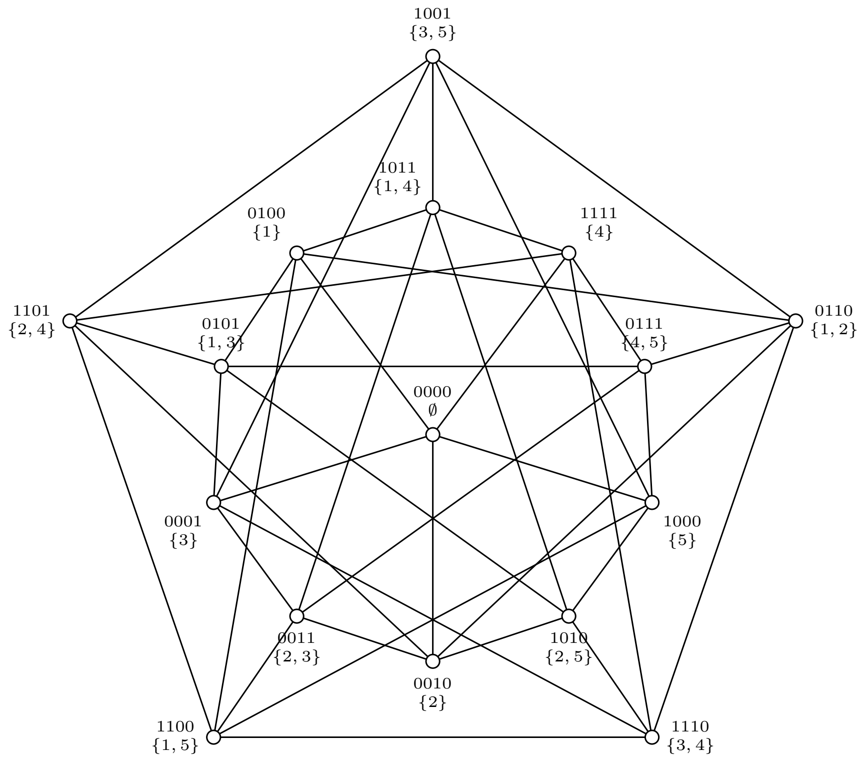

Figure 3.4: Clebsch graph $\square_5$

depicted skew 1-factor. In this fashion we get a geometrical description of a connection set over QD_{16}, corresponding to Q_4, as well as description of a skew 1-factor as a Cayley graph over QD_{16}. $\square$

Remark. The embedding of MK-graph into Q_4 used by us is briefly mentioned by Coxeter in [Cox50], p. 430. He however does not consider explicitly its group theoretical interpretation.

3.3 Clebsch graph

The Clebsch graph Cl is the unique strongly regular graph with the parameters $(v, k, l, \lambda, \mu) = (16, 5, 10, 0, 2)$. The name was coined by Seidel in [Sei68], following reference of Coxeter to Clebsch. Sometimes the name is attributed to the complementary graph. A nice proof of the uniqueness is presented in [GodR01]. A few models of Cl are known, cf. [BroCN89], [KliPR88].

Usually Cl is identified with the folded 5-cube. Therefore it is considered as a member of corresponding infinite series of distance regular graphs; this implies notation $\square_5$ for it which will be used by us from now on.

Two more models are depicted in Figure 3.4.

The upper labels of vertices refer to the Cayley graph over E_{2^4} with the connection set $X_5 = \{0001, 0010, 0100, 1000, 1111\}$. The lower labels refer

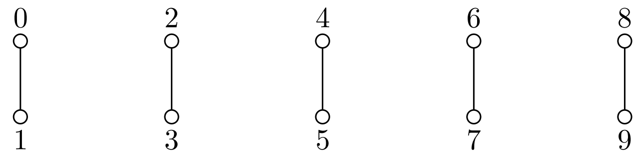

Figure 3.5: Auxiliary graph $5 \circ K_2$

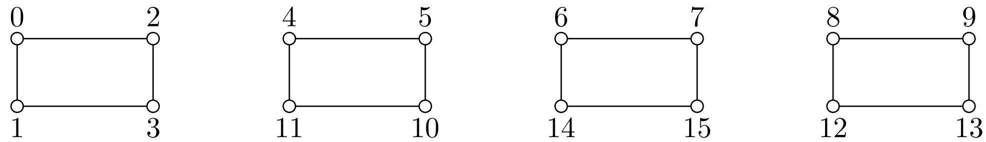

Figure 3.6: A skew system of quadrangles

to local model of graph with respect to special vertex $\emptyset$. The neighbors of $\emptyset$ are 1-element subsets of $[1,5]$, non-neighbors are 2-elements subsets of $[1,5]$ with evident adjacency between 1-sets and 2-sets. The fact that we have a desired strongly regular graph is visually observed from the diagram. As a Cayley graph, $\Box_5$ has a transitive automorphism group, while the local model reveals that stabilizer of a point is isomorphic to S_5. Finally we get that $G = Aut(\Box_5) \cong E_{2^4} \rtimes S_5$ is a rank 3 group.

The most famous description for the group G is irreducible Coxeter group $\mathfrak{D}_5$, see e.g. [GroB96]. A more naive though very helpful representation is related to auxiliary graph $5 \circ K_2$ as it is depicted in Figure 3.5.

Our group G in these terms may be interpreted as the subgroup $(S_5 \wr S_2)^{pos}$ of even permutations in $Aut(5 \circ K_2)$.

Graph $\Box_5$ has 705 1-factors in 7 orbits of length 5, 80, 60, 120, 240, 160, 40. Again we are interested in two smallest orbits.

The orbit of length 5 is evident. It consists of the five Cayley graphs over E_{2^4} with connection set $\{x\}$, $x \in X_5$. (We later on will call X_5 *frame.*) Removal of any such *direct 1-factor* from $\Box_5$ evidently provides a copy of Q_4.

Below we simultaneously consider one more labeling of vertices of $\Box_5$ borrowed from Q_4.

The orbit of length 40 is inherited from the skew 1-factors of Q_4. Clearly five copies of Q_4 inside of $\Box_5$ altogether produce 5×8 skew 1-factors. Note that each *skew 1-factor* of $\Box_5$ has a natural *mate* in form of the corresponding direct 1-factor. Easy inspection shows that two such 1-factors together provide a subgraph of $\Box_5$ of form $4 \circ C_4$, where C_4 is a quadrangle. An example of such structure, which will be called a *skew system of quadrangles* is depicted in Figure 3.6.

Finally we mention that $\Box_5$ has $\frac{5 \cdot 24}{3} = 40$ induced quadrangles and

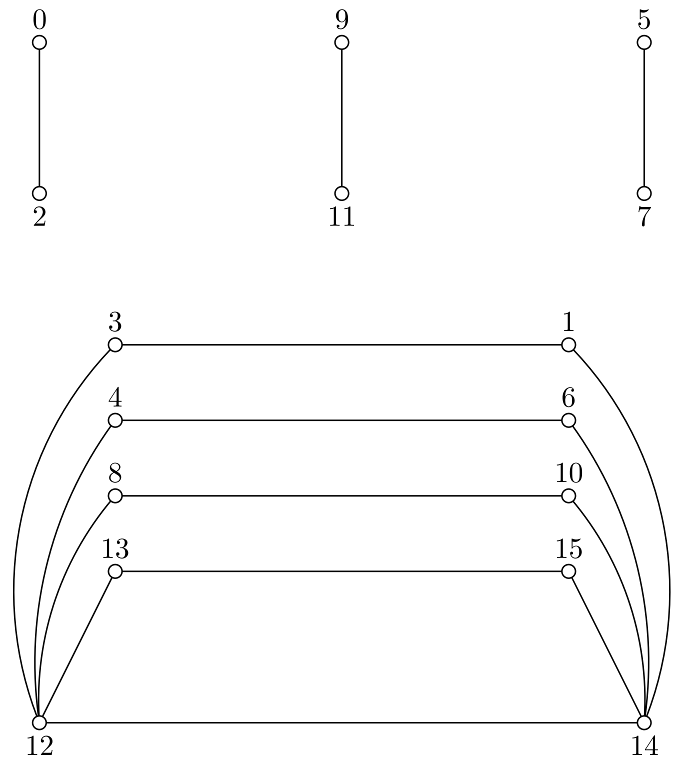

Figure 3.7: Edge decomposition of $\Box_5$

$\frac{16 \cdot 5}{2} = 40$ edges. G acts transitively on all three sets. We need to describe the stabilizer of representatives of the discussed structures.

The description of the stabilizer of a skew system of quadrangles follows from Section 3.2, this is group $K = \mathrm{G}L(2,3)$ in its transitive action.

It is also easy to see that the stabilizer of quadrangle is group $D_4 \times S_3$. (Indeed consider for example stabilizer of quadrangle $(0,1,5,4)$. Dihedral group D_4 acts diagonally on this quadrangle together with its "relatives" $(6,7,3,2)$, $(12,13,9,8)$, and $(14,15,11,10)$. Symmetric group S_3 leaves first quadrangle in place permuting the latter three quadrangles.

More tricky arguments are used in order to describe the stabilizer of edge, say $\{12,14\}$. A schematic diagram in Figure 3.7 shows edge $\{12,14\}$ together with their neighbors and separately the subgraph induced by the non-neighbors of the edge. All other edges are omitted from the diagram. Easy inspection reveals group $S_4 \times S_2$, acting on the bottom part of the diagram. It turns out that each permutation from this group may be extended to the automorphism of the upper part, showing visually isomorphism with the group $S_3 \wr S_2$ of automorphisms of octahedron.

Finally we get the following set of generators for the stabilizer of edge in

$\square_5$:

$$S_4 \times S_2 \cong S_3 \wr S_2 = \left\langle \begin{array}{c} (0,9,2,11)(1,6,10,15)(3,4,8,13)(5,7), \\ (0,7)(1,6)(2,5)(3,4), \\ (0,2)(1,3)(4,6)(5,7)(8,10)(9,11)(12,14)(13,15) \end{array} \right\rangle$$

We bring together requested information about the group G.

Proposition 3.3. *Let $G = Aut(\square_5)$ be the automorphism group of the Clebsch graph $\square_5$.*

a) $G \cong E_{2^4} \rtimes S_5 \cong (S_5 \wr S_2)^{pos} \cong \mathfrak{D}_5$ is a group of order 1920.

b) G acts transitively on each of the following systems:

- *40 quadrangles;*
- *40 edges;*
- *40 skew systems of quadrangles aka skew 1-factors.*

c) Stabilizer of quadrangle is $D_4 \times S_3$;
stabilizer of edge is $S_4 \times S_2 \cong S_3 \wr S_2$;
stabilizer of skew system is $K = \mathrm{GL}(2,3)$.

d) Third group in (c) acts transitively on the vertex set, while the first group has orbits of length 4 and 12, and the second group has orbits of length 2, 6 and 8.

e) The graph $\square_5$ can be considered as Cayley graph over groups E_{2^4} and QD_{16}.

Proof. $\square$

3.4 Cages

The notion of cage goes back to W. T. Tutte (see e.g. [Tut66]) who established foundation of the theory for a particular case of cubic graphs (regular graphs of valency 3).

According to [Sac63] for arbitrary $k \geq 3$ and $g \geq 3$ there exists at least one regular graph of valency k and girth g. A regular graph of valency k and girth g, and such that there are no smaller graphs with the same valency and girth is called a (k, g)-*cage* ([Big93]).

There is a natural lower bound for a number of vertices in a (k, g)-cage, commonly denoted by $n_0(k, g)$, which is formulated separately for odd and even girth (see [Big93]). Graphs which attain this bound are very rare (Moore graphs for g odd, and incidence (Levi) graphs of generalized polygons for g even).

Problem of description of (k, g)-cages is completely solved for a relatively small amount of values of (k, g).

An important characteristic feature of the classical cages such as Moore graphs and Levi graphs of generalized quadrangles is that they are coherent and moreover they are distance regular, therefore a coherent closure of such graph is a (metrical) association scheme.

In this context it is natural to expect that those cages which are also coherent are in a sense very close (from the point of view of algebraic graph theory) to the classical cages.

Cages of valency 3 are investigated with a reasonable success, all of them are known for girth at most ten, see e.g. [PisBMOG04].

The case of $(k, 3)$-cages is in a sense degenerate, these are complete graphs K_{k+1}.

Cages of girth 4 (projective planes) are classical objects of investigation in the area of finite geometries.

In this paper we will be slightly interested in the $(5, 4)$-cage.

Below we consider with more attention cages of girth 5. It is well known (see e.g. [CamL91]) that non-trivial Moore graphs may exist only for $g = 5$, and there are just 3 non-degenerate possibilities for the valency, namely $k = 3$, $k = 7$ or $k = 57$, leading to strongly regular graphs with $k^2 + 1$ vertices. The unique Moore graph of valency 3 is the Petersen graph, and the unique Moore graph of valency 7 is the Hoffman-Singleton graph. A question about the existence of a Moore graph of valency 57 is still open.

The cages of girth 5 and valency 3, 4, 5, 6, 7 have respectively 10, 19, 30, 40 and 50 vertices, all of them are nowadays completely classified. Below we consider valencies 6 and 7.

Following pioneering paper by C. W. Evans ([Eva79]) we consider in a given graph $\Gamma = (V, E)$ set $\mathfrak{S}_n$ of all cycles (circuits) of length n. Γ is called a *general net* if and only if there exists $\mathfrak{S}^* \subseteq \mathfrak{S}_n$ such that given any edge $e \in E$ there are exactly two cycles $C_1, C_2 \in \mathfrak{S}^*$ such that $e \in C_1$ and $e \in C_2$. In general the girth $g \leq n$; when $g = n$, Γ will be called a *general g net*. Moreover Γ is called a general g net cage of valency k if Γ is also a (k, g)-cage. An *embeddable net* may be drawn on a surface.

A number of net cages are investigated in [Eva79], including K_4, Cube, Petersen graph and Heawood graph for valency 3. A net of valency 6 and girth 5 on 40 vertices was constructed by Evans. At the time of publication of [Eva79] he was not aware precisely that this graph is a cage. We will consider it below.

3.5 Hoffman-Singleton graph

The unique strongly regular graph HoSi with the parameters $(50, 7, 42, 0, 1)$ was discovered in 1960 by A. J. Hoffman and R. R. Singleton [HofS60].

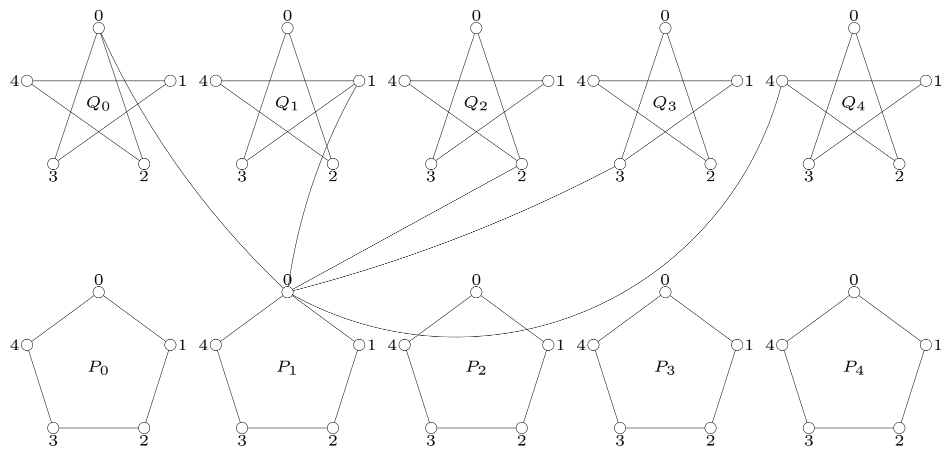

Figure 3.8: Robertson model for HoSi

Original proof of the uniqueness was already established in [HofS60] using beautiful linear algebra arguments.

N. Robertson in his Thesis [Rob69] suggested his famous pentagon-pentagram model which is repeated below. Higman in [Hig66] classified all rank 3 graphs on 50 points with subdegrees 1,7,42. Together with the proof of the uniqueness of HoSi his result implied description of the group $Aut(HoSi)$. Explicitly the description appeared in [BenL71] as group $P\Sigma U(3, 5^2)$ of order 252000.

A number of beautiful models of HoSi are known, most of them are based on the use of a certain maximal subgroup of $Aut(HoSi)$, in such fashion they are collected in the home page of A. Brouwer [Bro].

In this section we will briefly mention a few models which will be later on revised in a sense using in evident form concept of a coherent configuration.

Robertson model. The original description was purely pictorial (diagram 1.1c in page 12 of [Rob69]). It is repeated here in slightly modified form (cf. [BonM76]).

Here upper cycles are called *pentagrams* and low ones *pentagons*, with vertex i of P_j joined to vertex $i + jk \pmod 5$ of Q_k.

HoSi, the unique Moore graph of valency 7, contains as induced subgraphs, smaller Moore graphs of valency 2 and 3. Information about the equitable partitions, generated by pentagons and Petersen subgraphs, in principle may be extracted from the analysis of the Robertson model.

Following L. G. James [Jam74], see also [FanS93], let us consider a copy of pentagon P, set $N_1(P)$ of neighbors of some vertices in P which are not in P, and set $N_2(P)$ of vertices of distance 2 from P. We get an equitable partition of HoSi with sizes of cells 5, 25, 20.

A coherent configuration generated by this partition will be considered in Section 11.

Similarly, following [Jeu83], we realize that HoSi contains one orbit of

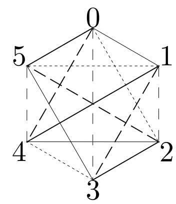

Figure 3.9: 1-factorization of K_6

size 525 consisting of induced Petersen graphs. Stabilizer of one such graph of order 480 will be considered with much detail in Sections 10, 11, see also Section 3.6 below.

A natural way to construct HoSi is to consider $Aut(HoSi)$ as rank 3 extension of S_7. This leads to a consideration of coherent configuration with 3 fibers of size 1, 7 and 42. Surprisingly, we did not find in literature an explicit presentation of this configuration. It will be considered below. First we need auxiliary construction, cf. [CamL91].

Proposition 3.4. *a) There are six different 1-factorizations of graph K_6, any two are isomorphic.*

b) The automorphism group of 1-factorization of K_6 is isomorphic to transitive action of S_5 on 6 points. This action is 3-transitive.

Proof. One of the factorizations is depicted in Figure 3.9. Let $g_1 = (0,1,2,3,4)$, $g_2 = (1,2,4,3)$, $g_3 = (0,1)(2,4)$. Let $H =< g_1, g_2, g_3 >$. It is clear that already $< g_1 >$ acts transitively on all five 1-factors, forming considered 1-factorization $\mathcal{F}$. Permutations g_2, g_3 also preserve $\mathcal{F}$. The whole group H acts transitively on 6 points, while $< g_1, g_2 >$ is a subgroup of order 20 in stabilizer of point 6 in H. Thus $|H| \geq 120$, therefore an orbit of $\mathcal{F}$ in S_6 has length at most 6. On other hand, easy to see that any two of 15 1-factors of K_6 belongs to exactly one 1-factorization. Thus, there are exactly 6 such objects, all of them belong to one orbit, and $|H| = 5!$, $H = Aut(\mathcal{F})$. □

Let $\Omega_1 = \{\emptyset\}$, $\Omega_2 = [0,6]$, $\Omega_3 = \mathcal{F}^{S_7}$, where $\mathcal{F}^{S_7}$ is orbit of $\mathcal{F}$ under action of S_7, provided that $\mathcal{F}$ is regarded as a system of subgraphs of K_7 with isolated vertex 6. Denote $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$. The symmetric group $S_7 = S([0,6])$ acts naturally on Ω with the orbits Ω_1, Ω_2, Ω_3. Thus we may consider a coherent configuration $\mathcal{H} = (\Omega, 2 - orb(S_7, \Omega))$.

Proposition 3.5. *a) $\mathcal{H}$ is rank 15 configuration with fibers of length 1, 7, 42.*

b) Representatives and lengths of basic relations are as presented in Table 3.1.

	Ω_1			Ω_2			Ω_3		
	#	valency	pair	#	valency	pair	#	valency	pair
Ω_1	0	1	$(0,0)$	1	7	$(0,1)$	2	42	$(0,8)$
Ω_2	3	1	$(1,0)$	4	1	$(1,1)$	6	36	$(1,8)$
				5	6	$(1,2)$	7	6	$(1,9)$
Ω_3	8	1	$(8,0)$	9	6	$(8,1)$	11	1	$(8,8)$
				10	1	$(8,7)$	12	30	$(8,9)$
							13	5	$(8,10)$
							14	6	$(8,17)$

Table 3.1: 2-orbits of action (S_7, Ω)

c) Merging of relations #1,3,7,10,14 provides a Moore graph on 50 vertices.

Proof. a),b) $\mathcal{H}$ has type $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{pmatrix}$. Indeed, taking into account the mode of action of S_7 on Ω_1 and Ω_2, we immediately justify all numbers between Ω_1 and other fibers and inside of Ω_2. Clearly, for point x from Ω_2 there are just two possibilities to be in relation with a 1-factorization from Ω_3, depending on $x = y$ or $x \neq y$, where y is isolated vertex of the 1-factorization.

Now, besides reflexive relation, we may distinguish 3 relations on Ω_3, described via typical representatives:

R_{12}: Isolated vertices of two factorizations are distinct, no pair of 1-factors from two factorizations, which share two edges.

R_{13}: Factorizations share the same isolated vertex.

R_{14}: Isolated vertices of two factorizations are distinct, any 1-factor from first factorization shares with a suitable 1-factor from second factorization two common edges.

For the reader's convenience list of all elements of Ω, as it is produced by GAP, is given in Supplement C.

We suggest the reader to check that the relations R_{12}, R_{13}, R_{14} are indeed 2-orbits of (S_7, Ω_3).

c) Let $\Gamma = (\Omega, E)$ be the graph with vertex set Ω and $E = R_1 \cup R_3 \cup R_7 \cup R_{10} \cup R_{14}$. The intersection diagram of Γ (with respect to the vertex from Ω_1) is depicted in Figure 3.10.

The correctness of diagram follows immediately from the description of basic relations of $\mathcal{H}$. To complete proof it is enough to justify similar diagram with respect to vertex from Ω_2 and from Ω_3. We leave this exercise to the reader. $\square$

Remarks.

a) Alternative way to complete proof of (c) is to submit a permutation which moves the vertex from Ω_1 and preserves graph Γ.

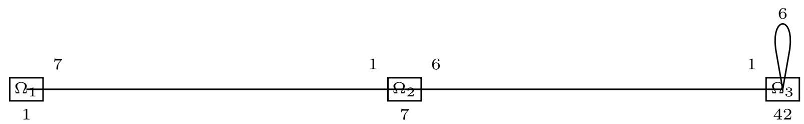

Figure 3.10: Intersection graph of graph Γ

b) The suggested model of $\Gamma = HoSi$ strictly exploits the exceptional property of the number six (see [CamL91]) related to the existence of two conjugacy classes of S_5 in S_6.

c) The graph (Ω_3, R_{14}) is a distance transitive graph of valency 6 on 42 vertices, which is an antipodal covering of K_7. Traditionally, it is described (see [BroCN89]) as the subgraph of the HoSi induced by its second constituent. Here we provide also its direct construction in terms of 42 1-factorizations of K_6 inside of K_7.

3.6 Anstee-Robertson graph

We now consider a regular graph $\mathcal{R}$ of valency 6 on 40 vertices as it was originally constructed by Neil Robertson in his thesis [Rob69].

Consider Robertson decomposition of HoSi, remove from it one pentagon and one pentagram, get the graph induced by the remaining 40 vertices. It is clear from the construction that the resulting graph $\mathcal{R}$ is regular of valency 6, contains cycles of length 5, and moreover, its girth is equal to 5. According to Robertson, $\mathcal{R}$ was the smallest graph of valency 6 and girth 5 available to him. A decade later on, this graph was rediscovered or characterized a few times, each time due to remarkable circumstances.

In [OKeW79] a different model of the graph was suggested. It was proved that any graph of valency 6 and girth 5 has at least 40 vertices, therefore the graph is $(5, 6)$-cage. The uniqueness of the $(5, 6)$-cage was proved in [Won79].

C. W. Evans constructed the same graph in his Ph.D thesis (1978) and presented it in [Eva79]. He proved also that this graph is the unique 5 net of valency 6 on 40 vertices, guessing that it is a cage.

Finally, R. P. Anstee in [Ans81] (the paper was submitted in 1978) found his own original way to the graph, presenting its adjacency matrix S as a solution of the equation

$$S^2 + S = J_{40} - A + 6I_{40}$$

where $A = A(10 \circ K_4)$. Moreover in presentation of Anstee the graph appears as a solution of a suggested analogue of group divisible designs for Moore graphs. Anstee was the first who considered the question about the structure of $Aut(\mathcal{R})$. He presented an outline of a proof that $Aut(\mathcal{R}) \cong \mathbb{Z}_4 \times S_5$ is a group of order 480.

It is clear from the Robertson model that $\mathcal{R}$ together with the Petersen graph P provides an equitable partition of HoSi. In fact, the stabilizer of a P inside of $Aut(HoSi)$ is the above group of order 480; this group was observed in [Jeu83]. The link between $\mathcal{R}$ and HoSi is also used in [Haf03].

Our interest in $\mathcal{R}$ was raised when we constructed input of Anstee model to GAP and obtained the same order 480 for the group $Aut(\mathcal{R})$, though of a different structure. Namely, $Aut(\mathcal{R}) \cong \mathbb{Z}_4.S_5$, where we get a non-split extension. As was correctly observed by Anstee, the quotient graph of $\mathcal{R}$ (with respect to the imprimitivity system, consisting of 10 disjoint independent sets of size 4) is isomorphic to $\overline{P}$. However, $Aut(P)$ is not embedded to $Aut(\mathcal{R})$ as Anstee was thinking. It seems that this delusion never was properly discussed in literature.

In fact, there are a few alternative ways to describe $Aut(\mathcal{R})$ as an abstract group, for example $2S_5.2$ (see [Bro]).

We discuss below a few remarkable models of $\mathcal{R}$, postponing more detail to Section 10, and exploiting extensively knowledge of the lattice of subgroups of $Aut(\mathcal{R})$ revealed with the aid of GAP.

<u>Robertson model.</u> The subgroup of $Aut(\mathcal{R})$ which preserves each of 8 cycles in Robertson decomposition is just group $\mathbb{Z}_5$ in a semiregular action.

<u>Anstee model.</u> The adjacency matrix S of the graph $\mathcal{R}$ discovered by Anstee has a block form, each of the blocks is a suitable circulant matrix of order 4. The subgroup of $Aut(\mathcal{R})$ which preserves each of 10 blocks in the Anstee model is $\mathbb{Z}_4$ in a semiregular action.

Remark. The visions of $\mathcal{R}$ presented by Robertson and Anstee are in a sense complementary or in other words orthogonal. In particular, the detected subgroups $\mathbb{Z}_4$ and $\mathbb{Z}_5$ are commuting, generating a semiregular subgroup $\mathbb{Z}_{20}$ in $Aut(\mathcal{R})$. This may serve us as a justification of the used name for the graph $\mathcal{R}$.

The model coined in [OKeW79] was also reproduced in [HolS93], p. 201. (Note that both diagrams on this page are relevant to the same model of $\mathcal{R}$.)

<u>Dihedral model.</u> $Aut(\mathcal{R})$ has a conjugacy class of length 6 consisting of dihedral groups D_{10} of order 20. Each such group acts on the vertex set with four orbits of length 10, leading to rank 88 coherent configuration J with 4 fibers of equal size and of the type

$$\begin{pmatrix} 6 & 6 & 5 & 5 \\ 6 & 6 & 5 & 5 \\ 5 & 5 & 6 & 6 \\ 5 & 5 & 6 & 6 \end{pmatrix}.$$

Note that here, $AAut(J) = CAut(J)/Aut(J)$ is a certain group of order 128.

One more remarkable model was suggested by Evans which in a sense amalgamates the models of Anstee and Robertson. We present its modification on Figure 3.11.

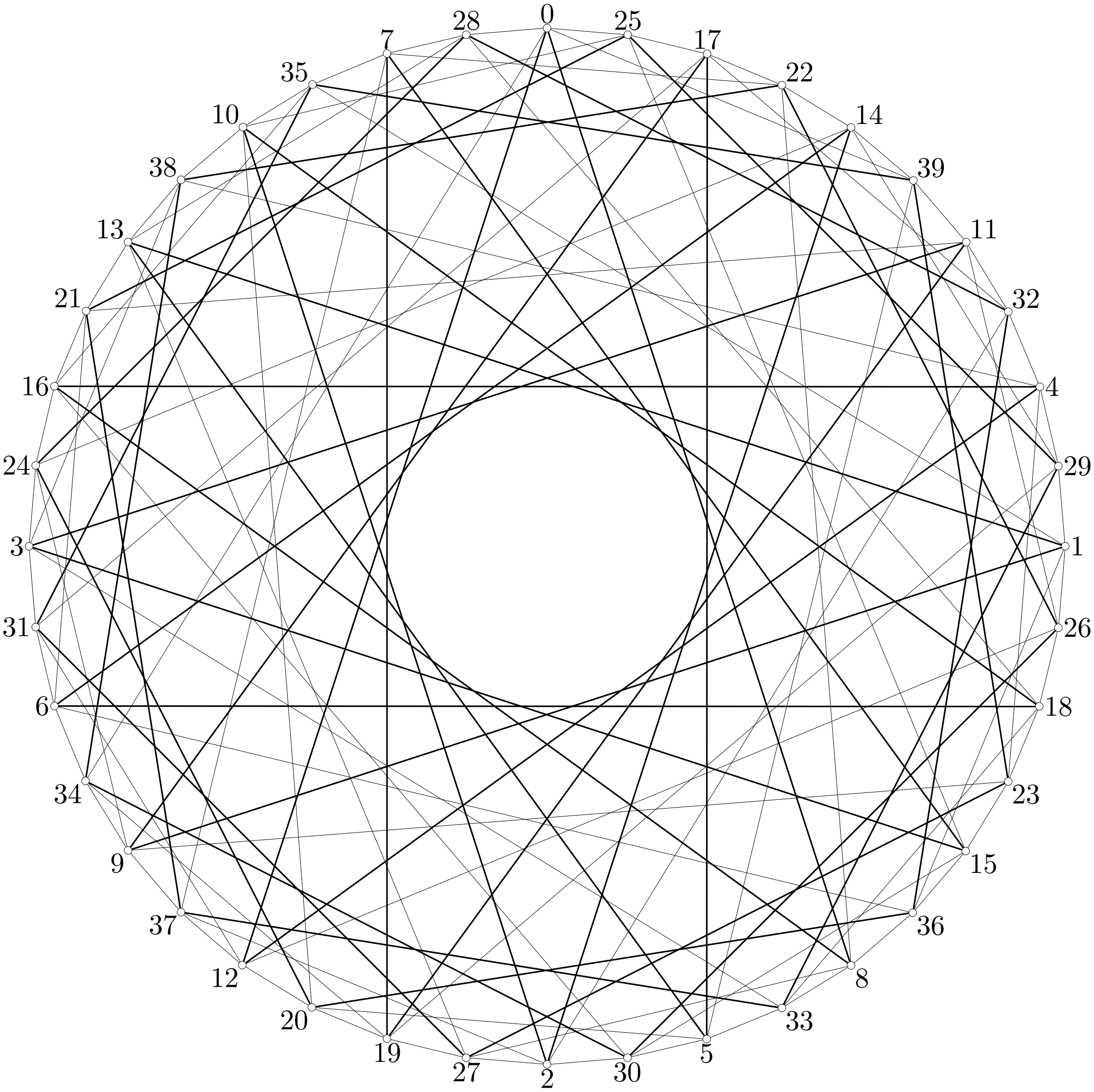

Figure 3.11: Hamiltonian model of Anstee-Robertson graph

Evans Hamiltonian model. We are working precisely with the Anstee model, labels of the vertices are shifted by -1 to the segment $[0, 39]$. The reader may distinguish on a diagram factorization of $\mathcal{R}$ into three regular spanning subgraphs of valency 2. First and second subgraphs are two disjoint Hamiltonian cycles, namely $(0, 25, 17, 22, 14, 39, 11, 32, 4, 29, 1, 26, 18, 23, 15, 36, 8, 33, 5, 30, 2, 27, 19, 20, 12, 37, 9, 34, 6, 31, 3, 24, 16, 21, 13, 38, 10, 35, 7, 28)$, $(0, 34, 19, 29, 14, 24, 9, 23, 4, 38, 3, 33, 18, 28, 13, 27, 8, 22, 7, 37, 2, 32, 17, 31, 12, 26, 11, 21, 6, 36, 1, 35, 16, 30, 15, 25, 10, 20, 5, 39)$.

The remainder (shown by bold lines) on the diagram is graph $8 \circ C_5$, consisting of 8 disjoint cycles of length 5. Four of these cycles are (up to numeration) Robertson pentagons, the other four are pentagrams. Namely, the cycles $P_0 = (20, 24, 28, 32, 36)$, $P_1 = (21, 25, 29, 33, 37)$, $P_2 = (22, 26, 30, 34, 38)$, $P_3 = (23, 27, 31, 35, 39)$ and $Q_0 = (0, 8, 16, 4, 12)$, $Q_1 = (1, 9, 17, 5, 13)$, $Q_2 = (2, 10, 18, 6, 14)$, $Q_3 = (3, 11, 19, 7, 15)$.

Let $g_1 = (0, 17, 14, 11, 4, 1, 18, 15, 8, 5, 2, 19, 12, 9, 6, 3, 16, 13, 10, 7)$ $(20, 37, 34, 31, 24, 21, 38, 35, 28, 25, 22, 39, 32, 29, 26, 23, 36, 33, 30, 27)$, $g_2 = (4, 16)(5, 17)(6, 18)(7, 19)(8, 12)(9, 13)(10, 14)(11, 15)(20, 22)(21, 23)$ $(24, 38)(25, 39)(26, 36)(27, 37)(28, 34)(29, 35)(30, 32)(31, 33)$.

The following observations may be easily confirmed visually:

- The group $\langle g_1^4 \rangle \cong \mathbb{Z}_5$ is a subgroup of the stabilizer of the Robertson decomposition.
- The group $\langle g_1^5 \rangle \cong \mathbb{Z}_4$ is a subgroup of the stabilizer of the Anstee decomposition.
- The group $\langle g_1 \rangle \cong \mathbb{Z}_{20}$ preserves both Hamiltonian cycles in the Evans model.
- Permutation g_2 exchanges the two Hamiltonian cycles.
- The group $\langle g_1^4, g_2 \rangle \cong D_5$.
- The group $H = \langle g_1, g_2 \rangle \cong \mathbb{Z}_4 \times D_5$ is group of order 40 acting with two orbits of length 20.

Remark. Existence of group H justifies in part origin of the "optical" group-theoretical illusion discussed above. This is the largest subgroup of $Aut(\mathcal{R})$ which may be presented as direct product of $\mathbb{Z}_4$ with a subgroup of the automorphism group of the quotient graph $\overline{P}$.

4 Total graph coherent configurations

Let $\Sigma = (V, E)$ be a graph. The *total graph* $T(\Sigma)$ is the graph with the vertex set $V \cup E$, two such vertices are adjacent in $T(\Sigma)$ if and only if they are adjacent or incident in Σ. (Here edges of Σ are incident if they have a joint vertex.) The concept of a total graph was suggested and investigated by M. Behzad et al, see e.g. [BehC66], [Beh70].

Example 4.1. $T(K_4)$

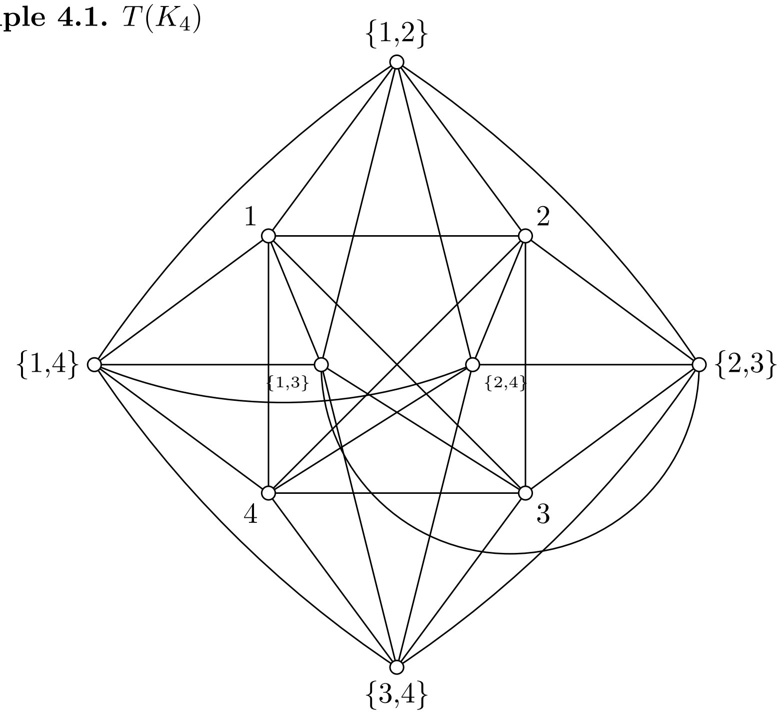

Figure 4.1: Total graph of graph K_4

Figure 4.1 depicts a diagram of the total graph of the complete graph K_4.

We will consider a coherent closure of $T(\Sigma)$ which will be called *total coherent configuration* of Σ. Our initial interest in this new concept was motivated by its possible application to the graph isomorphism problem, see Section 12.

Recall that in some exceptional cases the total graph $T(\Sigma)$ may have more rich automorphism group than original graph Σ. For example, while complete graph K_n, $n \geq 2$ has automorphism group S_n, the total graph $T(K_n)$ is isomorphic to the triangular graph $T(n+1)$, see [BehCN68]. Therefore $Aut(T(K_n)) \cong S_{n+1}$ is a proper overgroup of S_n.

In case $n = 4$ the fact of isomorphism $T(K_4) \cong T(5)$ may be easily observed from Figure 4.1. Indeed, after a substitution of vertex i by $\{i, 5\}$ for $1 \leq i \leq 4$ we recognize in diagram graph $T(5)$.

Looking for a non-trivial generalization of this simple observation we have arranged a number of computer algebra experiments with total graph coherent configurations for certain relatively small strongly regular graphs. Our goal was to detect in a total configuration for Σ a merging association scheme with a non-trivial overgroup of the group $Aut(\Sigma)$. Some of our results are discussed below.

Let us consider the total coherent configuration $\mathcal{T}(n)$ of the triangular graph $T(n)$ (recall that $T(n)$ is the line graph $L(K_n)$ of the complete graph K_n). In more detail we first construct the total graph $\mathbb{T}(n)$ of the triangular graph $T(n)$. The vertices of $\mathbb{T}(n)$ are the edges and the paths of length 2 of K_n, and two vertices of $\mathbb{T}(n)$ are adjacent if they both are edges of K_n with a common end, or else they have a common edge of K_n. Clearly $\mathbb{T}(n)$ has $\frac{n(n-1)}{2} + (n-2)\frac{n(n-1)}{2} = \frac{n(n-1)^2}{2}$ vertices.

While $\mathcal{T}(n)$ is simply the coherent closure of $\mathbb{T}(n)$, we also consider the *Schurian coherent closure* of $\mathbb{T}(n)$, denoted by $S(n)$. Here $S(n) = (\Omega, 2-orb(G,\Omega))$, a coherent configuration formed by the 2-orbits of the group $G = Aut(\mathbb{T}(n))$ acting on the vertex set Ω of $\mathbb{T}(n)$. It is well known that G is isomorphic to S_n for $n > 4$.

On this stage it is clear that $\mathcal{T}(n)$ is a certain merging of $S(n)$. Using COCO for small values of n we observed that rank and intersection numbers of $S(n)$ became in sense stable, starting from $n \geq 9$. This observation was confirmed for larger values of n.

Theorem 4.1. *Let $n \geq 9$, then:*

a) $S(n)$ has 2 fibers and 25 relations;

b) The intersection numbers as functions of n may be represented by a polynomial function on n of degree at most 3;

c) $S(n) = \mathcal{T}(n)$;

d) $S(n)$ has only two merging association schemes: rank 3 scheme with a basic graph isomorphic to $\frac{n(n-1)}{2} \circ K_{n-1}$, and rank 4 scheme isomorphic to wreath product $\mathfrak{m}(T(n)) \wr \mathfrak{m}(K_{n-1})$, where $\mathfrak{m}(T(n))$ is rank 3 scheme generated by $T(n)$, $\mathfrak{m}(K_{n-1})$ is a trivial scheme with one class on $n-1$ points.

Proof. Details are presented in [Ziv]. Currently the proof is computer dependent. Parts (a) and (b) are proved on a theoretical level. Using COCO we find intersection numbers of $S(n)$ for $n = 9, 10, 11, 12$ and after that determine polynomial expressions of intersection numbers with the aid of Lagrange interpolation. On the final stage, using symbolic computations with polynomials we arrange search for possible mergings, solving for each separate case suitable systems of Diophantine equations. $\square$

Remarks.

1. Part (c) of Theorem 4.1 is also valid for $4 \leq n \leq 8$. This was checked by combining computations with COCO and use of implementation of WL-closure algorithm.

2. For $n = 6, 7, 8$ $\mathcal{T}(n)$ has still 25 basic relations, while for $n = 5$ there are 24 basic relations.

3. Two standard mergings described above appear also for $5 \leq n \leq 8$. For $n \neq 5, 7$ there are no other (non-trivial) mergings.

4. For $n = 7$ we get a non-trivial exceptional merging which is of independent interest. It corresponds to an embedding of symmetric group of degree 7 to the group $U(4,3).2^2$ of order 13063680 and provides a new model for the unique Zara graph on 126 vertices, cf. [BloB84]. This case will be considered in [KliZJ].

5. For $n = 4$ we get rank 18 configuration which has altogether 7 merging association schemes of ranks 3 and 4. These schemes are, however, easily predictable, therefore we avoid detailed description of them.

It remains to pay a special attention to the case $n = 5$, which in fact was the origin of our interest in total graph configurations.

Example 4.2. *Total configuration $\mathcal{T}(5)$.*

We start from $S_5 = \langle (0,1,2,3,4), (0,1) \rangle$. Using COCO we get the following labeling of the set of vertices of $\mathbb{T}(5)$:

0	$\{0,1\}$	1	$\{1,2\}$	2	$\{2,3\}$	3	$\{0,2\}$
4	$\{3,4\}$	5	$\{1,3\}$	6	$\{0,4\}$	7	$\{2,4\}$
8	$\{0,3\}$	9	$\{1,4\}$	10	$\{\{0,1\},\{1,2\}\}$	11	$\{\{1,2\},\{2,3\}\}$
12	$\{\{0,1\},\{0,2\}\}$	13	$\{\{2,3\},\{3,4\}\}$	14	$\{\{0,2\},\{2,3\}\}$	15	$\{\{1,2\},\{1,3\}\}$
16	$\{\{0,4\},\{3,4\}\}$	17	$\{\{1,3\},\{3,4\}\}$	18	$\{\{2,3\},\{2,4\}\}$	19	$\{\{0,2\},\{0,3\}\}$
20	$\{\{0,1\},\{0,4\}\}$	21	$\{\{1,4\},\{3,4\}\}$	22	$\{\{0,4\},\{2,4\}\}$	23	$\{\{0,3\},\{3,4\}\}$
24	$\{\{1,3\},\{1,4\}\}$	25	$\{\{0,1\},\{1,4\}\}$	26	$\{\{0,2\},\{0,4\}\}$	27	$\{\{0,1\},\{0,3\}\}$
28	$\{\{1,4\},\{2,4\}\}$	29	$\{\{0,4\},\{1,4\}\}$	30	$\{\{0,2\},\{2,4\}\}$	31	$\{\{0,3\},\{0,4\}\}$
32	$\{\{0,2\},\{1,2\}\}$	33	$\{\{0,1\},\{1,3\}\}$	34	$\{\{1,2\},\{1,4\}\}$	35	$\{\{0,3\},\{1,3\}\}$
36	$\{\{1,2\},\{2,4\}\}$	37	$\{\{1,3\},\{2,3\}\}$	38	$\{\{0,3\},\{2,3\}\}$	39	$\{\{2,4\},\{3,4\}\}$

COCO returns a rank 24 coherent configuration $\mathcal{T}(5)$ with two fibers, $\Omega_1 = [0,9]$ and $\Omega_2 = [10,39]$. The following table describes representatives of 24 basic relations of $\mathcal{T}(5)$:

0	(0, 0)	1	(0, 1)	2	(0, 2)	3	(0, 10)
4	(0, 11)	5	(0, 13)	6	(0, 15)	7	(0, 29)
8	(10, 0)	9	(10, 2)	10	(10, 3)	11	(10, 4)
12	(10, 5)	13	(10, 10)	14	(10, 11)	15	(10, 12)
16	(10, 13)	17	(10, 14)	18	(10, 15)	19	(10, 17)
20	(10, 18)	21	(10, 22)	22	(10, 24)	23	(10, 28)

COCO also finds 9 non-trivial merging association schemes of $\mathcal{T}(5)$ as follows:

```
1. subscheme of rank 5 by merging (0,13)(2,5,11,6,12,22,19,20,14)*
(7,10,21)(1,3,8,15,18,16)(4,9,17,23)
2. subscheme of rank 5 by merging (0,13)(2,5,11,22,14)(7,10,21)*
(1,3,8,4,9,15,18,16,17,23)(6,12,19,20)
3. subscheme of rank 5 by merging (0,13)(2,7,10,3,8,22,19,20,16)*
(5,11,15)(1,6,12,21,18,14)(4,9,17,23)
4. subscheme of rank 5 by merging (0,13)(2,7,10,22,16)(5,11,15)*
(1,6,12,4,9,21,18,14,17,23)(3,8,19,20)
```

```
5. subscheme of rank 4 by merging (0,13)(2,5,11,6,12,22,19,20,14)*
(7,10,21)(1,3,8,4,9,15,18,16,17,23)
6. subscheme of rank 4 by merging (0,13)(2,7,10,3,8,22,19,20,16)*
(5,11,15)(1,6,12,4,9,21,18,14,17,23)
7. subscheme of rank 3 with parameters (40,3,2) by merging (0,13)*
(2,5,11,1,6,12,3,8,4,9,22,15,18,19,20,14,16,17,23)(7,10,21)
8. subscheme of rank 3 with parameters (40,12,2) by merging (0,13)*
(2,5,11,7,10,1,6,12,3,8,22,21,15,18,19,20,14,16)(4,9,17,23)
9. subscheme of rank 3 with parameters (40,3,2) by merging (0,13)*
(2,7,10,1,6,12,3,8,4,9,22,21,18,19,20,14,16,17,23)(5,11,15)
```

Finally, COCO returns orders and ranks of the automorphism group of each of the schemes, confirming that all nine schemes are Schurian.

merging	rank	$\|Aut\|$	subdegrees
1	5	1920	1,12,12,12,3
2	5	7680	1,24,6,6,3
3	5	1920	1,12,12,12,3
4	5	7680	1,24,6,6,3
5	4	$7608405715845120 = 2^{33}3^{11}5^1$	1,24,12,3
6	4	$7608405715845120 = 2^{33}3^{11}5^1$	1,24,12,3
7	3	$230078188847156428800 = 2^{38}3^{14}5^2 7^1$	1,36,3
8	3	51840	1,27,12
9	3	$230078188847156428800 = 2^{38}3^{14}5^2 7^1$	1,36,3

We are now making first attempt to recognize these schemes, using GAP where it is necessary.

Two isomorphic schemes of rank 3 correspond to wreath product $S_{10} \wr S_5$; one scheme with group of order 51840 has as class of valency 12 the point graph of a classical generalized quadrangle $Q(4,3)$. Two isomorphic schemes of rank 4 are copies of $\mathfrak{m}(T(5)) \wr \mathfrak{m}(K_4)$ as was mentioned above.

Thus we are faced with necessity to identify four rank 5 schemes which on this stage should be regarded as pleasant surprises. Indeed, those are sporadic examples which do not have their analogues for larger values of n.

Using GAP we immediately recognize that pairs of schemes having the same order of automorphism groups are isomorphic. We postpone to next sections consideration of the scheme with group of order 7680.

Proposition 4.2. *a) Total graph coherent configuration $\mathcal{T}(5)$ among other mergings has two isomorphic copies of association scheme $\mathfrak{m}$ with 4 classes of valency 12, 12, 12 and 3.*

b) $\mathfrak{m}$ is a Schurian association scheme. The group $Aut(\mathfrak{m})$ is rank 5 group of order 1920 isomorphic to $E_{16} \rtimes S_5$.

c) One of the classes of valency 12 in scheme $\mathfrak{m}$ represents the point graph of a generalized quadrangle of order 3.

d) This generalized quadrangle is isomorphic to $Q(4,3)$.

e) Another of classes of valency 12 corresponds to the direct product of Petersen graph with empty graph on 4 vertices.

f) Class of valency 3 represents imprimitive strongly regular graph $10 \circ K_4$.

g) The remaining class of valency 12 is connected graph of diameter 2 and girth 3 which generates the whole scheme $\mathfrak{m}$.

Proof. Parts of the current proof are based on the analysis of the results of computations with the aid of COCO and GAP. On this stage we wish just to give a computer free proof of part (c) which is of a certain independent interest, providing a non-standard model for generalized quadrangle of order 3 which turns out to be isomorphic to $Q(4,3)$.

According to presented above labeling the considered graph has set of directed arcs $R_4 \cup R_9 \cup R_{17} \cup R_{23}$. Relations R_4, R_9, as well as R_{17}, R_{23} are pairs of opposite antisymmetric 2-orbits. Thus, after symmetrization we get an undirected graph with vertex set $\Omega = \Omega_1 \cup \Omega_2$ and edge set $E = E_1 \cup E_2$, with the following typical representatives:
Ω_1: $\{a,b\}$, $|\Omega_1| = 10$;
Ω_2: $\{\{a,b\},\{b,c\}\}$, $|\Omega_2| = 30$;
E_1: $\{\{a,b\},\{\{b,c\},\{c,d\}\}\}$, $|E_1| = 120$;
E_2: $\{\{\{a,b\},\{b,c\}\},\{\{a,c\},\{c,d\}\}\}$, $|E_2| = 120$.

Here *typical* means that a,b,c,d,e represent pairwise distinct elements of the set $[0,4]$.

It is clear that graph $\Gamma = (\Omega, E)$ is a regular graph of valency 12 on 40 vertices. A simple inspection reveals that Γ has exactly 40 maximal cliques of size 4 which form one orbit under the action of the symmetric group S_5. A typical description of such clique is depicted in Figure 4.2.

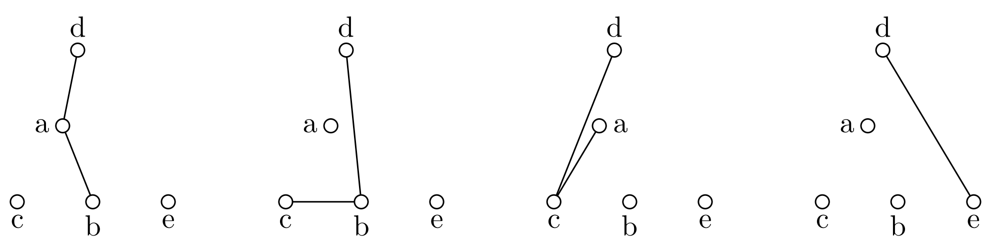

Figure 4.2: Line of $Q(4,3)$

Each such clique is uniquely determined by a diagram in Figure 4.3.

We have set L consisting of $5 \cdot 4 \cdot 2$ such possible diagrams which correspond to the 40 lines of a regular and uniform incidence structure $\mathfrak{S} = (\Omega, L)$. It remains to check that $\mathfrak{S}$ is indeed a generalized quadrangle. Note that stabilizer of a line of $\mathfrak{S}$ in group S_5 is a cyclic group of order 3

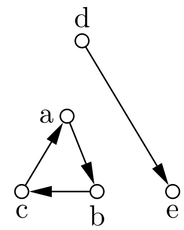

Figure 4.3: Structure corresponding to line of $Q(4,3)$

generated by (a,b,c). Group $\langle(a,b,c)\rangle$ has 14 orbits on Ω (4 on Ω_1 and 10 on Ω_2). 12 of those orbits correspond to the points not on a prescribed line. Routine inspection for a representative of each of these 12 orbits shows that there is exactly one line in $\mathfrak{S}$ through a considered point which intersects a prescribed line. $\square$

After discovering of exceptional mergings in $\mathbb{T}(5)$ we searched in literature and realized that part of results presented in Proposition 4.2 were known before. This directed our interest to a more systematic investigation of Higmanian association schemes, see next sections. Note however that the above model of $Q(4,3)$ seems to be new.

It was quite attractive to arrange hunting for other total coherent configurations admitting exceptional mergings. Because rank of such configurations drastically increases with the increasing of the rank of $Aut(\Sigma)$, there was evident sense to consider as initial objects certain rank 3 graphs.

The case $\Sigma = L_2(n)$ provides another well known classical series of rank 3 graphs, cf. [Sei67]. The total coherent configurations generated by these lattice square graphs were completely investigated in the same fashion as in Theorem 4.1 (see [Ziv]). However, this class of configurations does not provide surprises.

Another interesting option for Σ is provided by the Moore graphs.

The smallest degenerate Moore graph is pentagon. Clearly its total configuration contains the Petersen graph. The classical diagram, repeated below, represents vertices of pentagon as external cycle while vertices of the internal cycle correspond to edges of the external one (see [KliZJ] for more details).

The total configuration for Petersen graph does not contain mergings, while the total graph of its complement, the triangular graph $T(5)$ was presented in this section as origin of all project.

Similarly, the total graph configuration for the Hoffman-Singleton graph, $HoSi$ of valency 7 on 50 vertices does not have mergings at all. However, an exceptional merging for the total configuration of the complement, $\overline{HoSi}$ provides a very interesting primitive Schurian rank 5 association scheme on

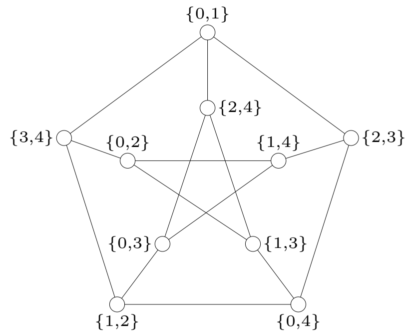

Figure 4.4: Diagram of the Petersen graph

1100 vertices. In our eyes this scheme provides a fresh alternative for consideration of a classical embedding of the automorphism group $Aut(HoSi)$ to the automorphism group of the Higman-Sims graph (see again [KliZJ] for details).

We postpone to the very end of our paper some speculations about the total configuration of the complement of a possible Moore graph of valency 57.

5 Rank 5 imprimitive association schemes: Higman's classification

In comparison with ranks 3 and 4, classification of rank 5 association schemes is still less developed. Note that 5 is the largest rank for which arbitrary scheme is commutative ([Hig75]). This, in principle, creates good opportunities for the use of classical feasibility conditions for commutative schemes. According to the use of such descriptors as symmetric and non-symmetric, primitive and imprimitive, metric or co-metric schemes, one may distinguish a large number of possibilities.

Information about metric schemes, that is distance regular graphs of diameter 4, may be found in [BroCN89]. Non-symmetric schemes are splittings of corresponding rank 3 or rank 4 schemes. It seems rather difficult to suggest a reasonable classification of all primitive rank 5 schemes.

Following [Hig95] we restrict ourselves to considering of symmetric imprimitive schemes of rank 5. Because 5 is prime, each decomposable scheme is wreath decomposable. We have exactly three possibilities for the rank and corank of a wreath decomposable scheme: $(3,3)$, $(2,4)$, $(4,2)$. Clearly the decomposition of such rank 5 scheme is reduced to the consideration of smaller ranks.

It was Higman who suggested in [Hig95] to consider the following three classes of symmetric imprimitive schemes containing a parabolic E:

Class of E	I	II	III
rank of E	3	2	2
corank of E	2	3	2

If quotient scheme for class II is imprimitive then the considered global scheme $\mathcal{H}$ has one more parabolic, E', which has rank 3. In this case, scheme $\mathcal{H}$ may be also attributed to class I. Clearly class III has empty intersection with classes I and II.

For each of three possible classes Higman provided in [Hig95] description of corresponding intersection matrices and character-multiplicity tables. There are interesting examples of schemes belonging to class I and to intersection of classes I and II, in particular those which appear from classical triality associated with groups $O_8^+(q)$ as well as sporadic examples associated with the groups $L_3(4)$, $U_6(2)$ and $U_3(5)$.

A family of instances of class III schemes which goes back to Ph.D thesis [Cha94], fulfilled under supervision of Higman, is also mentioned.

In what follows we will regard schemes belonging to class II but not to class I as *proper class II schemes*. Surprisingly, no one example of such scheme is mentioned in [Hig95]. Let us make a more accurate glance on this possibility.

Assume that $\mathcal{H}$ is an association scheme on set Ω consisting of nv points. Let E be an equivalence relation (parabolic) in $\mathcal{H}$ with n classes of size v. Let one of the classes of the quotient scheme on the set Ω/E be a strongly regular graph with n vertices and the parameters $k, l, \lambda, \mu, r, s, f, g$ in a traditional notation, here of course $n = 1 + k + l$. Let $\mathcal{H}$ be a proper class II scheme of rank 5.

Proposition 5.1. *a) The valencies of $\mathcal{H}$ have the form $v_0 = 1$, $v_1 = v - 1$, $v_2 = kS$, $v_3 = k(v - S)$, $v_4 = lv$.*

b) The character-multiplicity table is

$$\begin{array}{cc} \begin{pmatrix} 1 & v-1 & kS & k(v-S) & lv \\ 1 & v-1 & rS & r(v-S) & -(r+1)v \\ 1 & v-1 & sS & s(v-S) & -(s+1)v \\ 1 & -1 & x_1 & -x_1 & 0 \\ 1 & -1 & x_2 & -x_2 & 0 \end{pmatrix} & \begin{array}{c} 1 \\ f \\ g \\ z_1 \\ z_2 \end{array} \end{array}$$

here S is an extra parameter, $1 \le S < v$, x_1, x_2 are the roots of equation $x^2 - (\frac{rv}{S} - \lambda(v-S))x - \frac{kS(v-S)}{v-1} = 0$, $\tau = p_{33}^2$ one more extra parameter, and $z_1 = \frac{n(v-1)x_2}{x_2 - x_1}$, $z_2 = \frac{n(v-1)x_1}{x_1 - x_2}$, $z_1 + z_2 = n(v-1)$.

c) $0 < \mu < k$.

Proof. Formulas in parts (a) and (b) are given in [Hig95], we just improve a misprint in value for v_3 occurred on page 213. Part (c) follows immediately from the definition of proper class II scheme. $\square$

In fact, paper [Hig95] provides much more various helpful information, including intersection matrices, certain consequences of orthogonality relations, restrictions caused by the Krein conditions.

An interesting question is to find the smallest example (with respect to the number nv of vertices) of a proper class II scheme. Each symmetric association scheme on up to 15 vertices is Schurian, thus we have inspected the catalog [ConHM98], which reveals that no such scheme is available, see also Section 12.

To the best of our knowledge the scheme $\mathfrak{m}$ considered in proposition 4.2 is the only example of proper class II scheme which appears before in literature. For the first time in evident form this scheme was presented in [ChaH00] though its idea goes back to [DezD94]. Existence of this example implied our interest to a systematic consideration of all possible proper class II schemes of rank 5 on 40 vertices which forms the main target of this article.

Proposition 5.2. *a) There are just four following possibilities for the parameters of a rank 5 proper class II scheme on 40 vertices with* $v = 4$*:*

Case	k	l	S	v_0	v_1	v_2	v_3	v_4
a1	*6*	*3*	*2*	*1*	*3*	*12*	*12*	*12*
a2	*6*	*3*	*1*	*1*	*3*	*6*	*18*	*12*
a3	*6*	*3*	*2*	*1*	*3*	*6*	*6*	*24*
a4	*3*	*6*	*1*	*1*	*3*	*3*	*9*	*24*

b) There is no scheme corresponding to the class a4.

Proof. Part (a) follows immediately from the previous Proposition. Assume that there exists a scheme corresponding to the case a4. Let us consider its BM-algebra $\langle A_0, A_1, A_2, A_3, A_4 \rangle$. Here A_1 is the adjacency matrix of a parabolic relation. It is clear that $A_2^2 = 3A_0 + p_{22}^1 A_1 + p_{22}^2 A_2$. If $p_{22}^1 = 0$ then A_2 is also corresponding to a parabolic relation in contradiction to proper class II property.

If $p_{22}^1 \neq 0$ then p_{12}^2 is also non-zero. This implies that $p_{12}^3 = 0$ and $p_{12}^4 = 0$. We get that A_1 and A_2 correspond to parabolic, however this is impossible in principle, because $(1+3+3) \not| 40$. $\square$

In next sections we will completely characterize cases a1 and a2, and will prove existence of scheme corresponding to case a3.

6 Classical Deza graph on 40 vertices

In this section we consider with much more detail coherent configuration $\mathfrak{m}$ presented in Proposition 4.2. Our goal is to achieve a computer free interpretation of $\mathfrak{m}$ jointly with $Aut(\mathfrak{m})$ and to reveal its extra significant properties. Here $\mathfrak{m}$ represents example associated to case a1 in Proposition 5.2. The value of the extra parameter τ is equal to 4.

In fact, slightly routine computations taking into account restrictions on the parameters show that this is the only possible value in case a1.

6.1 Ridge graph

A notion of ridge graphs was introduced by A. Deza and M. Deza in [DezD94], see also [DezH03], [DezGP06]. A certain class of high dimensional polyhedra C_n was considered. The vertices of the ridge graph of a polyhedra C_n are the facets of C_n, two facets being adjacent if and only if their intersection is a face of codimension 2 of C_n.

According to [DezD94] with correction of misprint which appeared in [DezH03] the ridge graph $\overline{\Gamma_n}$ is regular with valency $\frac{2(n-3)(n^2-7)}{3}$ and any non-adjacent pair of vertices have either $\frac{2(n-3)(n^2-13)}{3}$ or $\frac{2(n-3)(n^2-16)}{3}+2$ common neighbors.

In our presentation the complement Γ_5 of the ridge graph $\overline{\Gamma_n}$ in case $n=5$ is of a particular interest. Up to the used notation the vertices of Γ_5 are 3-element subsets $\{i,j,k\}$ of $[1,5]$ together with pairs $(\{i,j\},k)$ (we are again using typical notation, so i,j,k represent different elements.) Altogether we have $10+30=40$ vertices.

Neighbors of $\{i,j,k\}$ are edges of the form $(\{i,j\},k)$ (3 such neighbors) and $(\{i,l\},j)$ (12 such neighbors).

Neighbors of $(\{i,j\},k)$ are $\{i,j,k\}$ (1), $\{i,k,l\}$ (4), $(\{i,k\},j)$ (2), $(\{i,k\},l)$ (4), $(\{i,l\},j)$ (4).

An easy inspection shows that the edge set of graph Γ_5 is the union of the edge sets of basic graphs of $\mathfrak{m}$ of valency 3 and 12 which are defined by the merging of the relations $(7,10,21,4,9,17,23)$ in the total graph configuration $\mathcal{T}(5)$. This confirms relevance of the graph Γ_5 and related structures to our considerations.

6.2 Deza graphs

Another important notion of Deza graph also goes back to [DezD94]. In evident form the concept was introduced in [EriFHHH99] as a generalization of strongly regular graphs. Namely a regular graph is a *Deza graph* if the number of common neighbors of two distinct vertices takes one of two values (not necessarily dependent on the adjacency of the two vertices).

As it is mentioned in [EriFHHH99], some Deza graphs may be constructed as a certain merging of classes in a symmetric association scheme. A simple criterion is formulated which allows to detect the mergings which lead to Deza graphs.

The notion of a Deza graph has also a natural formulation in terms of matrices. Suppose that Γ is an n-vertex graph with adjacency matrix M, while A and B are $(0,1)$-matrices such that $A+B+I=J$.

Then Γ is an (n, k, b, a) Deza graph if

$$(*) \qquad M^2 = aA + bB + kI$$

Note that Γ is strongly regular if and only if in $(*)$ we get that A or B is M.

According to this definition, both matrices A and B may be regarded as adjacency matrices of suitable graphs Γ_A and Γ_B. In this case, Γ_A and Γ_B are called *Deza children* (recall that [DezD94] was written by Antoine and Michel Deza). The cases when Deza children are strongly regular graphs are of a special interest. This is exactly the case on 40 vertices considered in the original paper [DezD94] and which is subject of our consideration. We will call this original Deza graph the *classical Deza graph*.

6.3 Deza family in a Higmanian house

We are now in a position to suggest an axiomatization for considered objects.

Definition 6.1. *Assume we have a Deza family on nv points consisting of Deza graph Γ and Deza children Γ_A and Γ_B with adjacency matrices M, A and B respectively. Assume that S is an adjacency matrix of disconnected graph $n \circ K_v$, such that $S + M + A' + B + I = J$, $S + A' = A$.*

Assume in addition that one of the matrices A, B, say B is adjacency matrix of a suitable strongly regular graph $\Delta = \Gamma_B$. If $\langle S, M, A', B, I\rangle$ is a symmetric Higmanian rank 5 association scheme of class II then the scheme will be called a Higmanian house for Deza family $(\Gamma, \Gamma_A, \Gamma_B)$.

Note that Deza family $(\Gamma, \Gamma_A, \Gamma_B)$ together with equivalence relation (*spread*) $n \circ K_v$ defines a rank 5 color graph which, in principle, may not correspond to an association scheme.

Making a short deviation we will consider a simple property of wreath product of association schemes.

Let $\mathcal{H} = (\Omega, \{R_0, R_1, \ldots, R_l, S_1, \ldots, S_m\})$ be a wreath product $\mathcal{H}_1 \wr \mathcal{H}_2$ of two association schemes with l and m classes as it was defined in Section 2. Assume that $A_1, \ldots, A_l, B_1, \ldots, B_m$ are the basic matrices corresponding to the classes of $\mathcal{H}$.

Proposition 6.1. *For $\mathcal{H} = \mathcal{H}_1 \wr \mathcal{H}_2$ (with notation as above) for each $1 \le i \le l$ and $1 \le j \le m$ is fulfilled*

$$A_i B_j = B_j A_i = v_j A_i,$$

where v_j is the valency of a basic graph Γ_j of $\mathcal{H}_2$.

Proof. If $((a, b), (e, f)) \in R_i$, then for any pair (c, d) such that $((a, b), (c, d)) \in R_i$ and $((c, d), (e, f)) \in S_j$, $c = e$, $(d, f) \in \Gamma_j$, therefore the amount of d is then valency of Γ_j.

On the other hand, if $((a, b), (c, d)) \in R_i$, $((c, d), (e, f)) \in S_j$ then $c = e$ and (a, c) is in basic relation i of $\mathcal{H}_1$, so the same is true for (a, e) and $((a, b), (e, f))$ is in R_i. $\square$

Proposition 6.2. *Let $\mathcal{H} = (\Omega, \{R_0, R_1, \dots, R_l, S_1\})$ be a wreath decomposable association scheme with equivalence relation $E = R_0 \cup S_1$ which is isomorphic to $\mathcal{H}_1 \wr \mathcal{H}_2$, where $\mathcal{H}_2$ is a scheme with one class defined by restriction of S_1 on any of equivalence classes of E. Assume that relation R_1 is split into a disjoint union of two symmetric relations T_1 and T_2.*

Assume also that

(a) there exists α such that $|T_1(x) \cap E(y)| = \alpha$ for each pair $(x, y) \in S_1$ and

(b) $(A(T_1))^2 \in \mathfrak{O} :=< A_0, A(T_1), A(T_2), A_2, \dots, A_l, B_1 >$,

then the color graph $(\Omega, \{R_0, T_1, T_2, R_2, \dots, R_l, S_1\})$ forms an association scheme.

Proof. By definition any product of two of matrices $A_0, A_1, \dots, A_l, B_1$ belongs to $\mathfrak{O}$. Let us prove that product of $A(T_1)$ and any of basic matrices also belongs to $\mathfrak{O}$. It is trivially true for A_0, and for $A(T_1)$ by condition (b). Condition (a) implies that $A(T_1) \cdot B_1 = (\alpha - 1)A(T_1) + \alpha A(T_2) \in \mathfrak{O}$.

Now for each $i \geq 2$ we get according to Proposition 6.1, that $A_i B_1 = v_1 A_i$. Therefore $(A_0 + B_1)A_i = (v_1 + 1)A_i$ or $A_i = \frac{1}{v_1+1}(A_0 + B_1)A_i$. Thus

$$\begin{aligned}
A(T_1)A_i &= \frac{1}{v_1+1}A(T_1)(A_0 + B_1)A_i = \\
&= \frac{1}{v_1+1}(A(T_1) + (\alpha - 1)A(T_1) + \alpha A(T_2))A_i = \\
&= \frac{1}{v_1+1}(\alpha A(T_1) + \alpha A(T_2))A_i = \\
&= \frac{1}{v_1+1}\alpha A_1 A_i \in \mathfrak{O}
\end{aligned}$$

Now because product of $A(T_1)$ and of A_1 with any of basic matrices of $\mathfrak{O}$ belongs to $\mathfrak{O}$, this is also true for product of $A(T_2) = A_1 - A(T_1)$, with the same matrices. $\square$

6.4 Master coherent configuration on 120 points

Let $G = Aut(\square_5)$ as it appears in Section 3.3. Let $K = \mathrm{G}L(2, 3)$ the automorphism group of a skew 1-factor (aka skew system of quadrangles) in $\square_5$, $Q = D_4 \times S_3$ the automorphism group of a quadrangle and $T = S_4 \times S_2$ the automorphism group of an edge in $\square_5$.

We consider Schurian coherent configuration $\mathfrak{n}$ which appears from the action of G on the cosets of Q, T and K in G. Clearly, this is a configuration on 120 points with 3 fibers of size 40. The configuration $\mathfrak{n}$ plays a central role in our presentation, in a sense it may be considered as a very weak analogue of configurations coming from triality (see Section 12).

According to geometric interpretation developed in Section 3.3 we prefer to consider points of $\mathfrak{n}$ as all possible quadrangles, edges and skew systems

fiber	quadrangles	edges	skew 1-factors
quadrangles	1,3,12,12,12	4,12,12,12	4,12,24
edges	4,12,12,12	1,3,4,8,24	8,8,24
skew 1-factors	4,12,24	8,8,24	1,3,4,8,24

Table 6.1: Valencies of relations in $\mathfrak{n}$

in $\square_5$. Initial information about $\mathfrak{n}$ may be easily attained with the aid of COCO.

Proposition 6.3. *a)* $\mathfrak{n}$ *is rank 35 configuration of type* $\begin{pmatrix} 5 & 4 & 3 \\ 4 & 5 & 3 \\ 3 & 3 & 5 \end{pmatrix}$ *with valencies of basic relations presented in Table 6.1;*

b) $Aut(\mathfrak{n}) = G$*;*

c) restriction of $\mathfrak{n}$ *on first fiber defines association scheme* $\mathfrak{m}$ *considered in Section 4;*

d) restriction of $\mathfrak{n}$ *on second or third fibers defines rank 5 schemes which are not algebraically isomorphic;*

e) rank 16 configuration on first and third fibers has a merging rank 5 scheme coming from $GQ(3)$*;*

f) rank 16 configuration on first and second fibers has two non Schurian merging schemes of rank 6 and 5.

Proof. □

Association schemes mentioned in (d) will be discussed in next sections. Below we provide computer free interpretations of a few results related to claims in Proposition 6.3.

Proposition 6.4. *Consider action of group* G *on the set* Ω *of 40 quadrangles of* $\square_5$*.*

a) $2-orb(G,\Omega) = (R_0, R_1, R_2, R_3, R_4, R_5)$*; explanation of relations is presented in Table 6.2.*

b) $R_0 \cup R_1$ *is an equivalence relation.*

c) R_2 *is an edge set of Deza graph.*

d) R_3 *is an edge set of a point graph of a generalized quadrangle.*

e) R_4 *is an edge set of wreath product of the Petersen graph with the empty graph* E_4 *on 4 vertices.*

No.	representative	invariants of representatives common vertices	common edges	skew or direct system	valency	name
0	$(0,1,5,4)$	4	4	n/a	1	Loops
1	$(2,3,7,6)$	0	0	direct	3	Spread
2	$(0,1,3,2)$	2	1	n/a	12	Deza
3	$(8,9,11,10)$	0	0	skew	12	$GQ(3)$
4	$(3,11,4,12)$	1	0	n/a	12	Petersen$[E_4]$

Table 6.2: 2-orbits of association scheme $\mathfrak{m}$

Proof. a) Let us consider again quadrangle $(0,1,5,4)$ as a reference one. Using methodology of description of 2-orbits of transitive permutation groups described in [FarKM94], we present in Table 6.2 five more representative quadrangles, showing for them number of common vertices, edges, and extra comments. The amount of quadrangles sharing with the representatives the same values of considered invariants follows from easy combinatorial counting. Thus it remains only to confirm by computation that group Q has in its induced action on Ω exactly 5 orbits.

b) Each quadrangle from $\square_5$ has a description inherited from Section 3.2. A slight modification of this description allows to identify a quadrangle with a coset of a suitable subgroup of order 4 in group E_{2^4} generated by two elements of the frame X_5 (as in Section 3.3), see Section 8 for more details. Then two quadrangles are in relation $R_0 \cup R_1$ if and only if they correspond to cosets of the same subgroup. (Equivalence classes of $R_0 \cup R_1$ will be called *direct systems* of disjoint quadrangles.)

Proof of (c) and (d) follows from next two propositions.

e) We now explain relation R_4 as follows: consider two disjoint 2-element subsets of X_5, construct corresponding subgroups of order 4, get cosets having (not empty) intersection. An easy exercise for the reader is to recognize in this description the same relation R_4 and to identify it as wreath product of the Petersen graph with empty graph E_4.

□

6.5 One more model of $Q(4,3)$

Let us define an incidence structure $\mathfrak{S}_1$ on two fibers of master coherent configuration $\mathfrak{n}$.

The points are quadrangles of $\square_5$, the lines are skew systems of quadrangles in $\square_5$, incidence is usual inclusion.

Proposition 6.5. *a)* $\mathfrak{S}_1$ *is a partial linear space with 40 lines of size 4 with point graph* Γ_3 *of valency 12.*

b) $\mathfrak{S}_1$ *is a generalized quadrangle of order 3.*

c) The structure $\mathfrak{S}_1^{\top}$ *dual to* $\mathfrak{S}_1$ *allows a spread.*

d) $\mathfrak{S}_1$ *is isomorphic to* $Q(4,3)$.

e) $Aut(\mathfrak{S}_1)$ *is a rank 3 group of order 51840.*

Proof. a) Follows from definitions.

b) We recall that G acts transitively on lines of $\mathfrak{S}_1$. Let us consider one of the lines, say $l = \{(0,1,3,2),(4,5,10,11),(6,7,15,14),(8,9,13,12)\}$. We know that $Aut(l) = L \cong GL(2,3)$. According to Table 6.1, K has three orbits on the point set Ω of length 4, 12, 24. First orbit clearly corresponds to the points of l. We get representatives $p_1 = (0,1,5,4)$ and $p_2 = (0,2,6,4)$ of two other orbits and find lines $l_1 = \{(0,1,5,4),(2,3,12,13),(6,7,15,14),(8,9,11,10)\}$ and $l_2 = \{(0,2,6,4),(1,5,10,14),(3,7,15,11),(8,9,13,12)\}$ which intersect l and contain the points p_1 and p_2 respectively. Simple counting of flags of $\mathfrak{S}_1$ shows that our partial linear space $\mathfrak{S}_1$ indeed satisfies the remaining axioms of generalized quadrangles of order 3.

c) Consider 10 direct systems of quadrangles and check that they indeed provide a spread in the dual structure $\mathfrak{S}_1^{\top}$.

d) There are exactly two $GQ(3)$ (up to isomorphism) which are dual to each other, see [Pay75], one of them $W(3)$ allows spreads, while other one, $Q(4,3)$ does not have spreads.

e) The automorphism group of both classical generalized quadrangles is known as group $P\Gamma U(4,2)$ of order 51840.

□

6.6 A new partial linear space

It remains to define one more incidence structure $\mathfrak{S}_2 = (\Omega, \mathcal{S})$ on the other fibers of $\mathfrak{n}$. Again the points are elements of the set Ω of quadrangles in $\square_5$ while lines are edges of $\square_5$. Incidence is dual to inclusion, that is edge is in a quadrangle.

Proposition 6.6. *a)* $\mathfrak{S}_2$ *is a symmetric incidence structure with 40 points and 40 lines of size 4.*

b) $\mathfrak{S}_2$ *is a partial linear space.*

c) For each line $l \in \mathcal{S}$ *there are precisely*

- *12 points $p \notin l$ through which there are no lines intersecting l.*
- *12 points $p \notin l$ through which there is exactly one line intersecting l.*
- *12 points $p \notin l$ through which there are exactly two lines intersecting l.*

Proof. The methodology of proof is quite similar to the previous Proposition. Again due to the transitivity of action of group G on $\mathcal{S}$ we select arbitrary reference edge, say $\{12, 14\}$, and observe that stabilizer T of the edge has on the set Ω orbits of length 4,12,12,12. We select representatives $(1, 14, 12, 3)$, $(0, 2, 3, 1)$, $(0, 4, 5, 1)$, $(0, 15, 14, 1)$, from each of the orbits and find lines $\{(0, 7, 15, 8), (1, 3, 12, 14), (2, 6, 9, 13), (4, 5, 10, 11)\}$, l,
$\{(0, 1, 5, 4), (2, 3, 12, 13), (6, 7, 15, 14), (8, 9, 11, 10)\}$,
$\{(0, 1, 14, 15), (2, 3, 11, 10), (4, 5, 7, 6), (8, 9, 13, 12)\}$,
which intersect the reference line and contain the shown representatives of orbits of T on Ω respectively. This proves (a), (b) and (c). $\square$

Let us consider again basic matrices of the considered scheme $\mathfrak{m}$, denoted as usually by A_i, $0 \leq i \leq 4$.

Proposition 6.7. *The following equation is satisfied for the basic matrices $A_0, \ldots, A_4$:*

$$(*) \qquad A_2^2 = 12A_0 + 2A_3 + 4(A_1 + A_2 + A_4)$$

Proof. The equation immediately follows from the computation of intersection numbers p_{22}^i by COCO. Alternative hand computation requires inspection of 14 paths of length 2 of quadrangles having a common edge (that is, adjacent in Γ_2.) List of such paths is presented in Supplement A. $\square$

Proposition 6.8. *a) Classical Deza graph is the point graph Γ_2 of $\mathfrak{S}_2$.*

b) $\mathfrak{S}_2$ is uniquely reconstructed from its point graph Γ_2.

c) $Aut(\mathfrak{S}_2) = Aut(\Gamma_2) = G$.

Proof. It follows from Proposition 6.7 that Γ_2 is Deza graph. In Proposition 6.5 graph Γ_3 was characterized as the point graph of a generalized quadrangle. These two facts together uniquely determine the classical Deza graph.

Each edge of $\square_5$ corresponds to a clique of size 4 in Γ_2. Thus we get at least 40 such cliques. A simple inspection of data provided in Supplement A shows that Γ_2 has exactly 40 cliques of size 4. This means that $\mathfrak{S}_2$ may be recovered from Γ_2 and therefore $Aut(\mathfrak{S}_2) = Aut(\Gamma_2)$ is a certain overgroup of the group G.

We now determine 2-orbits of action of G on the set S of lines in $\mathfrak{S}_2$. (The corresponding association scheme will be discussed in Section 11.5.)

Let us consider edge $\{0,1\} \in S$ as a reference point. Then the edges $\{1,5\}$, $\{4,5\}$, $\{13,15\}$, $\{10,14\}$ correspond to the 2-orbits with valencies 8, 4, 24 and 3 respectively.

It is easy to check that the edge $l_0 = \{0,1\}$ regarded as line of $\mathfrak{S}_2$ has intersection of size 1 with lines $l_1 = \{1,5\}$ and $l_2 = \{4,5\}$, while intersection with two other lines is empty. Each of three points on line l_1 not belonging to l_0 is incident with exactly one point on l_0. Similarly, each of three points on line l_2 is incident with zero points on l_0.

We now define an auxiliary graph Δ with vertex set S and edge set $\{\{0,1\},\{1,5\}\}^G$. Arguments presented above show that $Aut(\mathfrak{S}_2)$ acts faithfully on Δ as a subgroup of $Aut(\Delta)$. On other hand, it is clear that Δ is isomorphic to the line graph of $\square_5$. Using classical Whitney-Jung theorem (see e.g. [Har69]), we get that $Aut(\Delta) \cong Aut(\square_5) \cong G$. $\square$

We are now bringing together all information about the scheme $\mathfrak{m}$ and Deza graph Γ_2.

Proposition 6.9. *a) Association scheme* $(\Omega, 2-orb(G,\Omega))$ *represents the Schurian Higmanian association scheme* $\mathfrak{m}$.

b) The basic graph Γ_3 *is the point graph of generalized quadrangle* $Q(4,3)$.

c) The basic graph Γ_1 *is a spread in the structure dual to* $Q(4,3)$.

d) The basic graph Γ_2 *is the classical Deza graph.*

e) $\mathfrak{m}$ *is Higmanian house for the classical Deza family defined by* $\mathfrak{S}_2$.

f) Graph $(\Omega, \Gamma_1 \cup \Gamma_3)$ *is the ridge graph on 40 vertices.*

Proof. $\square$

7 A family of algebraically isomorphic association schemes on 40 vertices

A Schurian association scheme $\mathfrak{m}$ (called Higmanian house for a classical Deza family) was presented in previous sections from a few different points of view. We are now interested in classifying all association schemes which are algebraically isomorphic to $\mathfrak{m}$.

7.1 Algorithm and computer search

For the purpose of elaboration of algorithm, let us once more make a glance on $\mathfrak{m}$. Simple computations (with or without use of COCO) show that $\mathfrak{m}$ has exactly three non trivial mergings:

$$< R_0, R_1, R_2 \cup R_3, R_4 >, < R_0, R_1, R_2 \cup R_3 \cup R_4 >, < R_0, R_1 \cup R_2 \cup R_3, R_4 > .$$

First merging is a wreath product of schemes on 10 and 4 points, second merging corresponds to graph $10 \circ K_4$, the last merging has as a basic graph strongly regular graph of valency 12. This list of mergings implies that $\mathfrak{m} = \langle\langle R_2 \rangle\rangle$, in other words the coherent closure of the classical Deza graph Γ_2 coincides with $\mathfrak{m}$. It is important to notice that first and second mergings are uniquely determined (up to isomorphism) by their intersection numbers.

In this fashion the graph Γ_2 can be obtained as follows:

- Consider wreath product W=Petersen wr K_4, of the Petersen graph with K_4. This is a regular graph of valency 15;
- Find complement $\overline{W}$ of W;
- Remove from the complement $\overline{W}$ the point graph of $Q(4, 3)$.

Note that the structure of W is quite evident: there are 10 "hypervertices", that is complete graphs of size 4 and each "hyperedge" of W is a clique of size 8, consisting of two hypervertices + all edges joining vertices from one complete graph and the other. The quotient graph is Petersen graph.

Note also that in search for schemes algebraically isomorphic to $\mathfrak{m}$ we may substitute Γ_3 by arbitrary strongly regular graph of valency 12 on 40 vertices. All these graphs are known, there are precisely 28 non-isomorphic such graphs, see [Spe00].

The presented information justifies the following clear outline of an algorithm for the computer search of all association schemes algebraically isomorphic to $\mathfrak{m}$:

- For each strongly regular graph Γ in catalog [Spe00] consider its complement $\overline{\Gamma}$ of valency 27.
- Disregard $\overline{\Gamma}$ if it does not contain cliques of size 8. Otherwise describe all orbits of cliques of size 8 in $\overline{\Gamma}$.
- Use detected cliques as possible hyperedges in W=Petersen wr K_4. Classify all different (up to automorphisms from $Aut(\Gamma)$) possible embeddings of W into $\overline{\Gamma}$.
- Consider difference $\overline{\Gamma} \setminus W$ which is a regular graph of valency 12 (candidate to be an analogue of the classical Deza graph). Check that $\overline{\Gamma} \setminus W$ is Deza graph with Deza children Γ and $\overline{\Gamma}$.
- For each detected Deza graph $\overline{\Gamma} \setminus W$ find coherent closure. Disregard results if it has rank larger than 5.
- In case when coherent closure has rank 5 check whether it is algebraically isomorphic to the Higmanian scheme $\mathfrak{m}$.

Γ_i	$\mathfrak{M}_i$	$\lvert Aut(\Gamma)\rvert$	$\lvert orb(Aut(\Gamma))\rvert$	$\lvert Aut(\mathfrak{M})\rvert$	$\lvert orb(Aut(\mathfrak{M}))\rvert$	Geom.	4-cliques
Γ_1	1.1	48	$4, 12^3$	48	same	no	32
Γ_2	2.1	384	$16, 24$	384	same	no	8
Γ_2	2.2	384	$16, 24$	192	same	yes	40
Γ_3	3.1	8	$2^8, 4^2, 8^2$	8	same	no	20
Γ_4	4.1	12	$1, 3^3, 6^3, 12$	12	same	no	24
Γ_5	5.1	64	$8, 16^2$	64	same	yes	40
Γ_5	5.2	64	$8, 16^2$	32	$4^2, 8^2, 16$	no	24
Γ_6	6.1	51840	40	1920	same	yes	40
Γ_7	7.1	192	$4, 12, 24$	192	same	no	24
Γ_7	7.2	192	$4, 12, 24$	32	$4^2, 8^2, 16$	yes	40
Γ_8	8.1	8	$2^8, 4^2, 8^2$	8	same	no	28
Γ_9	9.1	48	$2, 4, 6, 12, 16$	16	$2^4, 4^4, 16$	no	32
Γ_{10}	10.1	16	$4^2, 8^4$	16	same	no	32
Γ_{10}	10.2	16	$4^2, 8^4$	8	$4^8, 8$	no	32
Γ_{11}	11.1	144	$4, 12, 24$	48	same	no	32

Table 7.1: Higmanian schemes

This algorithm was implemented in GAP with the aid of GRAPE and nauty.

In what follows we are using labeling of strongly regular graphs as in [Spe00]. It turns out that precisely first 11 graphs have cliques of size 8. All these graphs admit at least one *Higmanian association scheme*, that is scheme algebraically isomorphic to $\mathfrak{m}$. Altogether we get 15 schemes, the corresponding Deza graph in each scheme may be called *Higmanian (Deza) graph* because it completely defines the considered Higmanian scheme.

Of course, of a special interest are those Higmanian graphs which are also *geometric*, that is point graphs of a partial linear space which is an analogue of structure $\mathfrak{S}_2$ described in previous section. A necessary condition for this property is that a Higmanian graph contains at least 40 cliques of size 4.

The main results of computation are presented in Table 7.1. Here we show for each graph order of its automorphism group and lengths of its orbits on vertices. Similar information is provided for automorphism group of each association scheme.

Note that now the classical (Schurian) Higmanian association scheme $\mathfrak{m}$ coincides with the scheme $\mathfrak{m}_{6.1}$.

We provide also information about the amount of 4-cliques in each of 15 Higmanian graphs.

All 14 schemes besides the classical one are non-Schurian. An interesting correlation appears with the property to be geometric. Only those (4) Higmanian graphs are the point graphs of a suitable partial linear space which admit (like classical Higmanian graph) precisely 40 cliques of size 4.

All detected partial linear spaces have the same intersection property like the structure $\mathfrak{S}_2$.

Analysis of computational results shows that each time when we get a color graph containing a Deza graph, this graph defines an association scheme. In other words, the last steps of algorithm dealing with coherent closure seem to be redundant. This empirical observation is justified below.

Proposition 7.1. *Let $\Gamma = \Gamma_3 = (\Omega, R_3)$ be a strongly regular graph of valency 12, the complement of which allows an embedding of W. Let $\Gamma_1 =$*

$10 \circ K_4$ *be a disconnected graph induced by hypervertices of* W*. Assume that there exists* α *such that for each edge* $(x, y) \in R_3$*,* $|\Gamma_3(x) \cap \Gamma_1(y)| = \alpha$*. Then:*

a) Color graph $(\Omega, R_0, R_1, R_2, R_3, R_4)$ *where* $\Gamma_1 = (\Omega, R_1)$*,* $W = (\Omega, R_1 \cup R_4)$*, is an association scheme;*

b) The association scheme in (a) is algebraically isomorphic to $\mathfrak{m}$*.*

Proof. Using Proposition 6.2 together with our assumption, we get a proof of (a). Indeed, W corresponds to existence of a wreath decomposable initial association scheme. To prove (b) we mention that obtained scheme is proper class II scheme. According to parametrization of Higman, there is only one possibility (up to algebraic isomorphism), which corresponds to case a1 of Proposition 5.1. □

Remark. In fact, for all 15 schemes, $\alpha = 1$. Though in this case the possible modification of algorithm was not implemented in advance, in general, such improvement may be quite essential, allowing to make a more rigid inspection of candidates for embedding.

Two of detected Higmanian association schemes, namely $\mathfrak{m}_{2,1}$ and $\mathfrak{m}_{2,2}$ have reasonably large automorphism groups. This makes it possible to present for both schemes a computer free explanation. First, let us consider the strongly regular graph common for both schemes.

We consider again group $P = Aut(Q_4)$ as it was presented in Section 3.2. It is easy to observe that P has two orbits of length 24 and 16 on the set Ω of all quadrangles in $\Box_5$. First orbit, Ω_1, corresponds to all quadrangles inside of Q_4, while second orbit, Ω_2, involves those quadrangles in $\Box_5$ which have just two parallel edges in Q_4. Action of P on $\Omega_1 \cup \Omega_2$ defines a coherent configuration with two fibers.

Proposition 7.2. *Let* $\mathcal{X}_{2,1} = (\Omega, 2 - orb(P, \Omega))$ *be a coherent configuration defined by the action of* $P = Aut(Q_4)$ *on* Ω*. Then*

a) $\mathcal{X}_{2,1}$ *has two fibers* Ω_1 *and* Ω_2 *of length 24 and 16 corresponding to orbits of* (P, Ω)*.*

b) Rank of $\mathcal{X}_{2,1}$ *is equal to 16.*

c) Representatives of basic relations of $\mathcal{X}_{2,1}$ *are presented in Table 7.2.*

d) Mergings of relations 7,10,15,4 and 8,11,15,4 correspond to strongly regular graph of valency 12 with the automorphism groups P *and* G *of order 384 and 51840 respectively.*

e) $\mathcal{X}_{2,1}$ *allows two rank 5 association scheme* $\mathfrak{m}_{6.1}$ *and* $\mathfrak{m} = \mathfrak{m}_{2.1}$ *which appear from the mergings*
$(0, 12)(14, 5, 2)(13, 7, 10, 1)(8, 11, 15, 4)(6, 9, 3)$
and $(0, 12)(14, 5, 2)(8, 11, 13, 1)(7, 10, 15, 4)(6, 9, 3)$*.*
Here each bracket represents relation in each Higmanian scheme.

f) $CAut(\mathcal{X}_{2,1}) = Aut(\mathcal{X}_{2,1}) = (P, \Omega)$.

g) $AAut(\mathcal{X}_{2,1})$ *has order 2 and is represented by permutation* $t = (7,8)(10,11)$ *in action on the relations of* $\mathcal{X}_{2,1}$.

h) Rank 3 schemes generated by strongly regular graphs, $\mathfrak{m}_{2.1}$ *and* $\mathfrak{m}$ *are algebraic twins with respect to* $AAut(\mathcal{X}_{2,1})$

Proof. To prove (a) we have to consider just quadrangles which do not belong to Ω_1. It is evident that P acts transitively on the 32 edges of Q_4 as well as on 16 pairs of antipodal edges. Consider as a representative pair $\{\{3,11\},\{4,12\}\}$ of antipodal edges. This pair together with two edges $\{\{4,11\},\{3,12\}\}$ from the direct 1-factor $Cay(E_{2^4},\{1111\})$ forms a quadrangle of $\square_5$. Orbit of this quadrangle with respect to (P,Ω) has length 16 and forms fiber Ω_2. The results in (b)-(e) are obtained with the aid of COCO, while COCO-II was used to get (f)-(h).

We explain structure of relations moved by permutation t.

Relations 7 and 8 are between elements of Ω_1 and Ω_2, both have out-valency (in Ω_1) equal to 4 and in-valency (in Ω_2) equal to 6. Relation 10 is paired with 7, relation 11 with 8. It is enough to distinguish between the relations 7 and 8.

Let $A \in \Omega_1$ be a quadrangle in Q_4, say $A = (9,11,15,13)$. It appears in four systems of skew quadrangles. Let us take arbitrary one. It has one more quadrangle belonging to the same copy of Q_4, say $B = (0,1,5,4)$. They define partition of Q_4 into two disjoint Q_3. The corresponding skew 1-factor contains exactly two edges which do not touch vertices of A and B, in our case these are $\{7,8\}$ and $\{3,12\}$. These edges form a unique quadrangle C in the considered skew system which is associated to A in relation 7, here $C = (3,7,8,12)$.

Description of relation 8 is more simple. Consider again quadrangle A as a reference one. Select in it arbitrary edge, say $\{9,13\}$. Get the antipodal edge $\{2,6\}$. They together define unique quadrangle $D = (2,6,9,13)$ which is associated to A.

Now the strongly regular graph #2 with group (P,Ω) is obtained from the classical geometrical strongly regular graph #6 by simple switching: remove relations #7 and #10 and instead add relations #8 and #11. Quite routine arguments provide a proof that such switching leads to a strongly regular graph, though computer is better suited for a necessary inspection. $\square$

Group $P = Aut(Q_4)$ has three subgroups of index 2. We are interested in one of them, denoted by U, which is isomorphic to $E_{2^3} \rtimes S_4$. Recall that Q_4 is a bipartite graph with a bipartite partition of vertices $\{\{0,3,5,6,9,10,12,15\},\{1,2,4,7,8,11,13,14\}\}$. Group U is defined as the stabilizer of one class of this partition. As group acting on vertices of $\square_5$,

#	Fibres	Valency in Ω_1	Valency in Ω_2	Representative
0		1		(0,0)
1		8		(0,1)
2		2		(0,25)
3	Ω_1	4		(0,7)
4		8		(0,27)
5		1		(0,39)
6		8		(0,8)
7	Ω_1, Ω_2	4		(0,3)
8		4		(0,26)
9			12	(3,4)
10	Ω_2, Ω_1		6	(3,0)
11			6	(3,16)
12			1	(3,3)
13			6	(3,6)
14	Ω_2		3	(3,18)
15			6	(3,19)

Table 7.2: Description of basic relations of $\mathcal{X}_{2,1}$

U has two orbits of length 8, namely those sets forming the partition. We consider induced action of U on the same set Ω of quadrangles in $\square_5$.

Proposition 7.3. *Let* $\mathcal{X}_{2,2} = (\Omega, 2 - orb(U, \Omega))$.

a) $\mathcal{X}_{2,2}$ *is a coherent configuration of rank 20 with the same fibers* Ω_1, Ω_2 *as in Proposition 7.2.*

b) $Aut(\mathcal{X}_{2,2}) = U$, $CAut(\mathcal{X}_{2,2})$ *is a group of order 1152,* $CAut(\mathcal{X}_{2,2})/Aut(\mathcal{X}_{2,2}) \cong S_3$.

c) $AAut(\mathcal{X}_{2,2}) \cong S_4$, *therefore* $\mathcal{X}_{2,2}$ *has proper algebraic automorphisms.*

d) Among algebraic mergings of $\mathcal{X}_{2,2}$ *there are two non-Schurian coherent configurations with fibers* Ω_1, Ω_2 *of rank 16 and 12 with the automorphism groups* U *and* P *respectively.*

e) $\mathcal{X}_{2,2}$ *has 31 non-trivial merging association schemes, 12 of them are rank 5 Higmanian association schemes of type a1. 6 of these schemes are isomorphic to* $\mathfrak{m}_{2.2}$, *3 of them to* $\mathfrak{m}_{2.1}$, *and 3 to* $\mathfrak{m}_{6.1} = \mathfrak{m}$.

f) All above 12 schemes form one orbit under $AAut(\mathcal{X}_{2,2})$.

Proof. The results were obtained with the aid of COCO and COCO-II. We intend to present a computer free explanation of interesting parts of the results in a subsequent publication. $\square$

Coherent configuration $\mathcal{X}_{2,2}$ is clearly a splitting of $\mathcal{X}_{2,1}$. While increasing of the rank (from 16 to 20) is not so essential, corresponding algebraic group is becoming much more rich.

In next two sections we will go on in a sense to the very end, splitting our classical rank 5 scheme to a coherent configuration of high rank. This will imply an explosion of the size of the algebraic automorphism group. Paradoxically, we will be able to reflect more transparent understanding of resulted huge objects in comparison with currently available vision of $\mathcal{X}_{2,2}$.

8 WFDF coherent configurations

In this section we will describe a general class of coherent configurations with many fibers, which were recently introduced by the author M.M., following ideas of Wallis and Fon-Der-Flaas. A specific example of such configurations will be investigated in the next section.

8.1 Initial notions

We briefly introduce two main ingredients of our future general constructions.

As first (internal) ingredient we will consider complete affine amorphic association scheme (as it was introduced in [GolIK85]). Recall that each such association scheme has n^2 vertices, $n+1$ classes and bijectively corresponds to an affine plane of order n. Namely, vertices of scheme correspond to the points of an affine plane. Each parallel class of affine lines is associated with a class of scheme, which is the edge set of a disconnected graph isomorphic to $n \circ K_n$ (disjoint union of n copies of complete graph of order n).

Conversely, any partition of edges of complete graph K_{n^2} into $n+1$ copies of graphs of form $n \circ K_n$ clearly depicts an affine plane of order n or alternatively a complete affine association scheme on n^2 vertices.

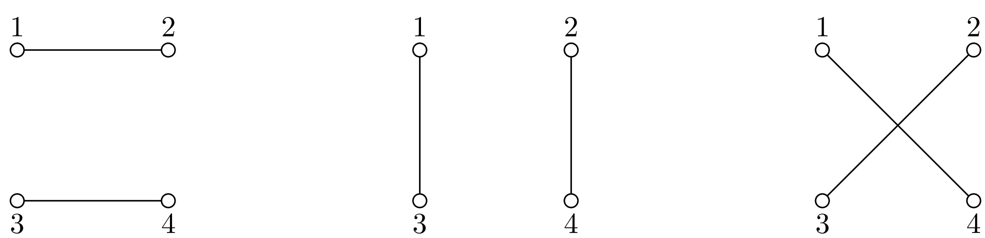

Figure 8.1: Parallel classes

Example 8.1. *Figure 8.1 depicts 3 parallel classes of affine plane of order 2.*

A partial linear space is an incidence structure $\mathfrak{S} = (\mathcal{P}, \mathcal{B})$ such that any two distinct points are incident to at most one block, and all blocks (also called lines) have size of at least 2. In a *uniform* partial linear space all lines have the same size. Uniform partial linear spaces will play a role of second (external) ingredient in our coming definitions.

Example 8.2. *a) A regular graph of valency k with n vertices introduces a degenerate partial linear space. Here each block is represented by an edge.*

b) $S(2, k, v)$, that is a Steiner system *whose blocks have size k is a very significant example of a partial linear space, which is a* linear space *(that is, any two distinct points appear in exactly one block.)*

Example 8.3. *We consider the complement of the Petersen graph (see Figure 4.4) as it was depicted in Figure 4.1, that is the triangular graph $T(5)$. This graph is geometrical. Indeed, let us consider its maximal (with respect to inclusion) cliques of the form $\{\{i, j\}, \{i, k\}, \{j, k\}\}$, where $\{i, j, k\} \subseteq [1, 5]$ is a subset of size 3. Clearly, all 10 cliques of such form represent blocks of a uniform partial linear space with 10 points coinciding with the vertices of $T(5)$.*

We are now in a position to define WFDF coherent configurations as a kind of a "blow up" of an external structure, namely of a partial linear space. Each point of the space will be substituted by a copy of an affine plane of order n. The points of such affine planes will form the fibers of the resulting configuration.

8.2 Main definitions

Definition 8.1. *Let $\mathfrak{O}_i = (V_i, \{C_{i,1}, \ldots, C_{i,n+1}\})$ be a copy of affine plane of order n. Here, $|V_i| = n^2$, for $1 \leq j \leq n+1$. $C_{i,j}$ is a graph of form $n \circ K_n$, which usually will be regarded as a parallel class $\#_j$. We will assume that a certain labeling of parallel classes of $\mathfrak{O}_i$ is prescribed. We need m copies of planes $\mathfrak{O}_i$ which are labeled by elements from $[1, m]$. Let $\mathfrak{S}$ be a partial linear space of order m (that is with m points), assume that each point of $\mathfrak{S}$ is incident with at most $n + 1$ lines of $\mathfrak{S}$.*

Let us consider as a new vertex set the set $V = V_1 \cup V_2 \cup \cdots \cup V_m$, here we assume that vertex sets V_i, $1 \leq i \leq m$ are pairwise disjoint. Sometimes it is convenient to consider V as Cartesian product $V = [1, n^2] \times [1, m]$, then each set V_i is attributed to $[1, n^2] \times \{i\}$.

We now define an arc partition of the complete graph $K_{n^2 \cdot m}$ with the vertex set V, that is a complete color graph with $n^2 \cdot m$ vertices.

Each set V_i is playing a role of a fiber in a coming coherent configuration. Inside of this fiber we naturally get $n + 2$ colors, that is relations $C_{i,j}$, $1 \leq j \leq n + 1$, and also the identity relation Δ_i on V_i.

Now we have to define relations between different fibers.

Let us first consider the Levi (incidence) graph $L(\mathfrak{S})$ of partial linear space $\mathfrak{S} = ([1,m], \mathcal{H})$. For each point $i \in [1,m]$ we label all blocks from $\mathcal{H}$ incident to i by distinct elements of $[1, n+1]$ (this is always possible due to our assumption on $\mathfrak{S}$.) A used labeling will be denoted by f_i.

Now we take an arbitrary block (hyperedge) $h \in \mathcal{H}$. Let $i,j \in [1,m]$ be incident to h. Assume that f_i assigns s_i to (i,h), while f_j assigns s_j to (j,h). Then we take class C_{s_i} from fiber V_i and class C_{s_j} from fiber V_j. Recall that each such class can be regarded as a partition of $[1,n^2]$ into n subsets of cardinality n. We now consider bijection σ_{ij} between the two partitions associated to the classes C_{s_i} and C_{s_j} (a question about possible restrictions for such bijections will be discussed below).

Note that a bijection σ_{ij} completely defines σ_{ji}, and vice versa.

With the aid of σ_{ij} we define a directed regular bipartite graph R_{ij} of valency n: each vertex from class x of a partition C_{s_i} is joined by an arc with each vertex from class $x^{\sigma_{ij}}$ in partition C_{s_j}. We also define $\overline{R_{ij}}$ as a complement to R_{ij}, that is $\overline{R_{ij}} = (V_i \times V_j) \setminus R_{ij}$.

Note that if i,j are not joined by a suitable hyperedge from $\mathcal{H}$ then R_{ij} is empty, that is between the fibers V_i and V_j we get just one relation $\overline{R_{ij}} = V_i \times V_j$.

It remains to describe additional requirements to the bijections σ_{ij}. This depends on cardinality of $h \in \mathcal{H}$. If $|h| = 2$ there is no extra requirements. Otherwise let $|h| \geq 3$, and $i,j,k \in h$. Then we claim that

$$(*) \quad \sigma_{ij} \cdot \sigma_{jk} = \sigma_{ik}.$$

(Practically restriction $(*)$ means that we may select arbitrary bijections from a certain element $i \in h$ to all other $j \in h$. After that the remaining bijections are uniquely determined by $(*)$.)

Proposition 8.1. *a) The resulted complete colored graph $\mathfrak{m} = (V,R)$ is defined correctly and has $n^2 \cdot m$ vertices.*

b) Assume that each point of $\mathfrak{S}$ is collinear with k other points. Then the graph $\mathfrak{m}$ has $m(n+m+k+1)$ colors, m of them are reflexive.

Proof. Correctness follows from the property of a partial linear space: any two points are joined by at most one line. Clearly we have $m(n+2)$ colors inside of the fibers, m of them are reflexive, and $2mk + m(m-1-k)$ colors between the fibers. □

Note that we have a lot of "degrees of freedom" in our definition: Selection of functions f_i and σ_{ij}. Also in principle, it is not necessary that the affine planes, attributed to distinct fibers, should be isomorphic.

Proposition 8.2. *Color graph $\mathfrak{C} = (V,R)$ defines a coherent configuration.*

Proof. Let W be a vector space spanned by the adjacency matrices of all relations from R. Fulfillment of all axioms of a coherent algebra, besides one is quite evident. Thus the only non-trivial part of a job is to check that product of any two basic matrices belongs to W. We consider a few different cases:

a) Both relations are from the same fiber: Use Lemma 3.2 from [GolIK85].

b) Both relations are between the same two distinct fibers i, j: Use properties of imprimitive strongly regular graph.

c) First relation is between fibers i and j, second relation is between fibers j, k where i, j, k are pairwise distinct.

   1) i, j, k are not collinear. Then $A(R_{ij})A(R_{jk}) = J_{ik}$.
   2) i, j, k are collinear. Then $A(R_{ij})A(R_{jk}) = nA(R_{ik})$.

d) First relation is between fibers i and j, second relation is between fibers k and l, i, j, k, l are pairwise disjoint. In this case the product of two adjacency matrices is clearly equal to zero matrix, thus belonging to W.

□

8.3 Generalizations and particular cases

The author M.M. was using in [Muz], see also [KliMPWZ07] a term half-homogeneous coherent configuration for a more general case in comparison with the one defined in previous subsection. In what follows we will use the term *WFDF configuration* for these objects.

A WFDF configuration which is defined with the aid of an affine plane of order n and projective plane of order n will be called a WFDF configuration of *type* $AP(n)$. A detailed investigation of various properties of the unique configuration $AP(2)$ is presented in [KliMPWZ07].

A WFDF configuration which is defined with the aid of an affine plane of order n and complete graph K_{n+2} (regarded as linear space with blocks of size 2) will be called a WFDF configuration of *type* $AK(n)$. Investigation of the unique $AK(2)$ was announced in [KliMR05].

Note that the construction suggested in [Muz] is of a more general nature: in a role of incidence structure, instead of an affine plane, an arbitrary affine resolvable 2-design (in sense of [Fla02]) may be considered.

In what follows we will restrict our attention by a consideration of a very particular WFDF configuration on 40 points for which affine plane of order 2 serves as internal structure and a partial linear space arising from the graph $T(5)$, see Example 8.3, is the external structure.

Note that though in most interesting for us examples of WFDF configurations we were using uniform external structure, this requirement to a partial linear space is not necessary in the given definition.

9 WFDF configuration on 40 vertices and some of its mergings

Basing on our recent experience of consideration of WFDF configurations with 16 and 28 vertices, we started to speculate that some of the Higmanian association schemes on 40 points presented in Section 7 may be obtained in a unified manner from a suitable WFDF configuration on 40 points. To confirm such guess we started from the classical Higmanian association scheme $\mathfrak{m}$ on 40 points, colored the 10 cells of the parabolic relation E with 10 distinct colors and constructed coherent closure of the resulted colored graph. As a result we obtained a rank 190 coherent algebra W with 10 fibers, which turns out to be Schurian. Using some experimental programs from COCO-II we described groups $Aut(W)$, $CAut(W)$, $CAut(W)/Aut(W)$ and $AAut(W)$. Afterwards we have managed to get a computer free interpretation of a major part of our discoveries. They are presented below.

Model of W. We start with group $H = E_{2^4}$ which we regard as vector space in dimension 4 over $\mathbb{G}F(2)$. Let $\{e_1, e_2, e_3, e_4\}$ be the standard basis of H, consider the set $X_5 = \{e_1, e_2, e_3, e_4, e_1 + e_2 + e_3 + e_4\}$, which will be called a *frame* in H. Clearly, 5 vectors of this frame are linearly dependent, while any 4 of them are independent. Now we consider any two distinct vectors from frame, generate a subgroup of order 4, and construct all cosets of such subgroup in H. In such manner we get a set Ω which consists of 40 cosets. Group H acts faithfully on Ω by shifts. The proof of the following proposition is straightforward.

Proposition 9.1. *Let $W' = (\Omega, 2 - orb(H, \Omega))$.*

a) W' is a rank 190 Schurian coherent configuration with 10 fibers of size 4. Each fiber corresponds to 4 cosets of the same subgroup.

b) Association scheme induced by each fiber is isomorphic to complete affine scheme of order 2 as it is presented in Example 8.1.

c) Let Ω_i and Ω_j be two fibers generated by disjoint 2-element subsets of X_5. Then all arcs starting in Ω_i and ending in Ω_j form one relation from $2 - orb(H, \Omega)$.

d) Let Ω_i and Ω_j be two fibers generated by 2-element subsets having intersection of cardinality one. Then $2 - orb(H, \Omega)$ contains exactly two relations formed by arcs starting in Ω_i and ending in Ω_j.

e) $W' \cong W$ and forms a WFDF coherent configuration with internal structure isomorphic to affine plane of order 2 and external structure isomorphic to one presented in Example 8.3.

Proof. As a control sum the reader may count $10 \cdot 4 + 30 \cdot 1 + 60 \cdot 2 = 190$ relations of kind (b), (c), (d). □

In what follows we will identify coherent configurations W and W'.

First we wish to describe the 2-closure of (H, Ω). For an arbitrary permutation group (G, Ω) we consider (as was suggested in [KalK76]) an invariant $f = f(G, \Omega)$ of (G, Ω) which is called the *degree of freedom* of (G, Ω). Here by definition f is the smallest amount of elements from Ω such that their pointwise stabilizer in G is the identity group.

Proposition 9.2. *Group (H, Ω) is 2-closed. In other words, $Aut(W) = H$.*

Proof. According to definition of W, $Aut(W)$ consists of all the automorphisms of $\mathfrak{m}$ which preseve each of the cells of the parabolic relation $10 \circ K_4$ in $\mathfrak{m}$. Recall that $Aut(\mathfrak{m})$ coincides with the group G of order 1920. Group H is a normal subgroup of group G and it is maximal normal 2-subgroup of G (acting intransitively on Ω). Thus $H = Aut(W)$. $\square$

Lemma 9.3. *Let (G, Ω) be intransitive permutation group with orbits $\Omega_1, \ldots, \Omega_k$. Assume that G does not act faithfully on any of orbits Ω_i. Then the centralizer $K = C_{Sym(\Omega)}(G)$ coincides with the direct sum of the centralizers $C_{Sym(\Omega_i)}(G, \Omega_i)$ of each separate action.*

Proof. Clearly for each $1 \leq i \leq k$ we get $C_{Sym(\Omega_i)}(G, \Omega_i) \leq K$. Therefore also the direct sum $C_{Sym(\Omega_1)}(G, \Omega_1) \oplus \cdots \oplus C_{Sym(\Omega_k)}(G, \Omega_k)$ is a subgroup of K. Assume that K contains more permutations besides this direct sum. Then at least two orbits of group G are merged to an orbit Ω' of group K. Get contradiction with Proposition 4.3 from [Wie64] which implies that action (G, Ω') is semiregular. $\square$

Proposition 9.4. *The centralizer $C_{Sym(\Omega)}(H)$ of (H, Ω) is an elementary Abelian group of order 2^{20}.*

Proof. The action of group H on each of its 10 orbits is not faithful and coincides with regular group E_4. The centralizer of each single regular group E_4 coincides with itself. Now use Lemma 9.3. $\square$

Proposition 9.5. *a) $CAut(W) = N_{Sym\Omega)}(H)$.*

b) $CAut(W) \cong \left(S_5, \left\{ {[1,5] \atop 2} \right\}\right) \wr (E_4, E_4) \cong E_{2^{20}} \rtimes S_5$, a group of order $2^{23} \cdot 3 \cdot 5$.

Proof. a) For arbitrary Schurian coherent configuration its color group coincides with the normalizer of the automorphism group in corresponding symmetric group. Now use Proposition 9.2.

b) By simple properties of wreath product, the group $\left(S_5, \left\{ {[1,5] \atop 2} \right\}\right) \wr (E_4, E_4)$ is a semidirect product of $E_{2^{20}}$ with S_5, therefore S_5 normalizes $E_{2^{20}}$, which in turn coincides with group $C_{Sym\Omega)}(H)$. Due to Proposition 9.1(e), the group $CAut(W)$ acts on the fibers as a subgroup of the automorphism group of the point graph of the external structure which is known to be isomorphic to $T(5)$. Because $\left(S_5, \left\{ {[1,5] \atop 2} \right\}\right) = Aut(T(5))$, we get that $CAut(W) \cong E_{2^{20}} \rtimes L$, where $L \leq S_5$. Now straightforward analysis of the structure of $W = W'$ confirms that $L = S_5$.

□

Corollary 9.6. *$CAut(W)/Aut(W)$ is a group of order $2^{19} \cdot 3 \cdot 5$.*

Proposition 9.7. *a) As an abstract group $AAut(W)$ is isomorphic to the same group $E_{2^{20}} \rtimes S_5$;*

b) $CAut(W)/Aut(W)$ is a non-normal subgroup of index 16 in $AAut(W)$.

Proof. Both parts were confirmed using computer package COCO-II and GAP. Here we provide a brief outline of a possible computer free proof.

First we need to count structure constants of W. For this purpose it is enough to distinguish two main cases: 3 fibers correspond to a line of the external structure, or correspond to a non-collinear triple of the points of the external structure.

Let $\{i, j, k\}$ be a line of the external structure, $\Omega_i, \Omega_j, \Omega_k$ are corresponding fibers. Then, from one of these fibers (say, Ω_i) to any other (say, Ω_j) there are two directed relations in W. Assume τ_{ij} is a permutation (on relations of W) which permutes these two relations as well as the opposite pair of relations. Then the group $\langle \tau_{ij}, \tau_{ik}, \tau_{jk} \rangle$ is elementary Abelian group of order 8. Denote by $H_{i,j,k}$ subgroup of index 2, namely $H_{i,j,k} = \langle \tau_{ij}\tau_{ik}, \tau_{ij}\tau_{jk}, \tau_{ik}\tau_{jk} \rangle$.

Check, using knowledge of the structure constants, that all permutations in $H_{i,j,k}$ are algebraic automorphisms. Now, using similar subgroups of $AAut(W)$ for each of the 10 lines of external structure, we show that a subgroup of order 2^{20}, isomorphic to $(E_{2^2})^{10} = E_{2^{20}}$ preserves all fibers and is a subgroup of $AAut(W)$.

Again, using the knowledge of the structure constants we get that no permutation of the form τ_{ij} is in $AAut(W)$. Thus we may justify that $E_{2^{20}}$ is the whole kernel of action of $AAut(W)$. We already know from Corollary 9.6 that the action of $AAut(W)$ is at least S_5. Clearly, it can not be larger because, by our definitions this action has to preserve the external structure.

To prove (b) it is enough to check that a permutation $\tau_{ij}\tau_{ik}$ does not normalize $CAut(W)/Aut(W)$. □

Remarks.

1. In what follows we will call permutations of the form $\tau_{ij}\tau_{ik}$ *elementary algebraic automorphisms* of W. Because these permutations do not belong to $CAut(W)/Aut(W)$ they are *proper algebraic automorphisms*, that is those which are not induced by a suitable permutation from $CAut(W)$ acting on points of W.

2. It follows from our description of $AAut(W)$ that this group acts faithfully and transitively on the 120 relations between two fibers corresponding to adjacent ordered pairs of vertices of graph $T(5)$. On two other sets of 2-orbits of size 40 and 30 group $E_{2^{20}}$ acts trivially, thus the action of $AAut(W)$ coincides with S_5. Moreover, to describe action of $AAut(W)$ it is enough to consider instead of set of "ordered" relations of size 120, the set of "non-ordered" relations of size 60.

3. In what follows we need to represent some computer data related to W and $AAut(W)$. Because we identify W and W' it is convenient to use description of points of W in terms of classical scheme $\mathfrak{m}$ which in turn (see Section 6) is inherited from the Clebsch graph in canonical representation.

Coding of W. We consider the normal subgroup E_{2^4} of the $Aut(\Box_5)$ of order 16 in its action on the vertices of the $\Box_5$ as follows:

$$E_{2^4} = \left\langle \begin{matrix} (0,1)(2,3)(4,5)(6,7)(8,9)(10,11)(12,13)(14,15) \\ (0,2)(1,3)(4,6)(5,7)(8,10)(9,11)(12,14)(13,15) \\ (0,4)(1,5)(2,6)(3,7)(8,12)(9,13)(10,14)(11,15) \\ (0,8)(1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15) \end{matrix} \right\rangle$$

We consider the computer generated list of all quadrangles of $\Box_5$ as vertices of $\mathfrak{m}$ and W (see Supplement A), denoted by Ω.

We get list of all elements generating E_{2^4} in the action on Ω.

Using COCO we obtain list of representatives of all 190 basic relations from W (see Supplement B). Using COCO-II, we get list of generators of $CAut(W)$ as permutations on Ω and induced permutations on the set R of relations of W. Finally we get list of generators of the group $(AAut(W), R)$. This data is used in the course of computer proof of the next Proposition.

$G = Aut(\Box_5)$ in action on 40 squares, is a subgroup, of $CAut(W)$, its action on 190 colors of W coincides with a group, S, of order 120, isomorphic to S_5, considered as a subgroup of $AAut(W)$.

Proposition 9.8. *a) Group $A = AAut(W)$ has 2^{20} subgroups conjugate to S in A.*

b) These groups split into three conjugacy classes with respect to the action of subgroup $Q = CAut(W)/Aut(W)$ by conjugation on subgroups of A, the cardinalities of these classes are 65536, 655360, 327680.

c) Representatives of these conjugacy classes lead to 3 association schemes which are algebraically isomorphic to $\mathfrak{m}$ but are not combinatorially isomorphic. The schemes are 2.1, 6.1, 7.1 according to enumeration in Section 7.

d) *In fact, these three schemes are algebraic twins, namely two non-Schurian schemes are obtained from the classical Schurian one with the aid of action of suitable algebraic automorphisms from A.*

Proof. In our eyes, this proposition provides a nice example of a mathematical claim justified strictly with the aid of a computer. All corresponding data is provided in Supplement B. Using this data, the reader may in principle, arrange an independent computer verification of our results. Three representatives for the mergings which give schemes isomorphic to 6.1, 2.1 and 7.1 are listed in Supplement B, as well as algebraic automorphisms mapping scheme 6.1 to 2.1 and 7.1. □

Remark. We checked that the automorphism groups of the remaining 12 Higmanian schemes from the family described in Section 7 do not contain subgroup $H = E_{2^4}$ and thus these schemes can not appear as algebraic mergings of any subalgebra of the coherent algebra W. Thus in a sense the above 3 schemes should be regarded as exceptional ones.

10 Coherent closure of cage on 40 vertices

10.1 Computer aided results

In what follows we consider the same copy of the graph $\mathcal{R}$ as it was depicted in Section 3.6. Denote for this section $G = Aut(\mathcal{R})$.

Using GAP we obtain that G is group of order 480.

Using the set of generators of G as input to COCO, we get that G in action on the vertex set Ω of $\mathcal{R}$ is a transitive group of rank 7 with the subdegrees 1, 2, 1, 12, 6, 12, 6 and representatives of 2-orbits $(0,0)$, $(0,1)$, $(0,2)$, $(0,4)$, $(0,8)$, $(0,9)$ and $(0,10)$ respectively.

COCO returns 7 non-trivial merging schemes of the symmetric association scheme $\mathcal{X} = (\Omega, 2-orb(G,\Omega))$, among them the scheme $\mathfrak{m}_{a2} = (\Omega, \{R_0, R_1 \cup R_2, R_3, R_4, R_5 \cup R_6\})$. Analysis of all the merging schemes of $\mathcal{X}$ shows that $\mathfrak{m}_{a2} = \langle\langle \Gamma_4 \rangle\rangle$, that is the coherent closure of the graph Γ_4 coincides with $\mathfrak{m}_{a2}$. COCO shows that $Aut(\mathfrak{m}_{a2}) = G$, thus $\mathfrak{m}_{a2}$ is a non-Schurian association scheme. Because $\{0,8\}$ is an edge of $\mathcal{R}$, we understand that $\mathcal{R} = \Gamma_4$. Finally, analyzing intersection numbers of $\mathfrak{m}_{a2}$ calculated with the aid of COCO, we conclude that $\mathfrak{m}_{a2}$ is a proper class II Higmanian association scheme which represents case a2 in formulation of Proposition 5.2 (here $\tau = 6$).

Using GAP once more, we conclude that $G = (SL(2,5) : \mathbb{Z}_2) : \mathbb{Z}_2$. Computations with GAP show that G has three subgroups of index 2, all of structure $SL(2,5) : \mathbb{Z}_2$, which will be denoted by K, L, M; $N = SL(2,5)$ is the unique subgroup of order 120 in G. We also determine the lattice of conjugacy classes of G consisting of 76 classes.

We now wish to interpret (as much as it is possible) the obtained results without essential use of a computer. We will use the fact that $\mathcal{R}$ is cage on 40 vertices which is unique up to isomorphism.

10.2 A few subgroups of G

Proposition 10.1. *a) Group G has an imprimitivity system S consisting of 10 blocks of size 4. Each block of system induces an empty subgraph of size 4. There is only one such system in G.*

b) Stabilizer of all blocks of system S in G is a cyclic group $\mathbb{Z}_4$ acting semiregularily on the vertex set Ω.

c) The quotient graph $\mathcal{R}/S$ of the graph $\mathcal{R}$ with respect to S is the complement $\overline{P}$ of the Petersen graph.

d) $|Aut(\mathcal{R})| \leq 480$.

Proof. This proposition is a summary of correct results inherited from [Ans81]. System S is formed by blocks of order 4 forming the matrix given by Anstee. A group $< g_1^5 >$ is a cyclic group $\mathbb{Z}_4$ which stabilizes each block of the system S. Arguments provided in p. 19 of [Ans81] justify that Z_4 is the full stabilizer of the blocks of S. Uniqueness of S is also claimed in [Ans81]. Thus $Aut(\mathcal{R}) \cong \mathbb{Z}_4.Y$, where Y is a suitable subgroup of S_5. □

Now we first wish to provide a computer free proof of the fact that $Aut(\mathcal{R})$ acts transitively on Ω. Recall that we already are aware of a subgroup $H \leq Aut(\mathcal{R})$, $H \cong \mathbb{Z}_4 \times D_5 =< g_1, g_2 >$ as in Section 3.6.

Proposition 10.2. *a) There exists a subgroup $Q \leq Aut(\mathcal{R})$ of order 80 which acts transitively on Ω.*

b) $Q \cong \mathbb{Z}_4.AGL(1,5)$.

c) Q contains a regular subgroup R.

Proof. Let us consider the following partition
$\tau = \{\{P_0, P_1, P_2, P_3\}, \{Q_0, Q_1, Q_2, Q_3\}\}$ of the Robertson decomposition of $\mathcal{R}$ (according to the labels of cycles presented in Section 3.6). Group H acts on the 8 cycles of this decomposition as E_4, preserving each of two cells of the decomposition and having two orbits of size 20 in the action on the set Ω.

Let us now consider subset $A \subseteq \Omega$, where A is the union of first and fifth cell in Anstee decomposition, namely $A = \{0, 1, 2, 3, 20, 21, 22, 23\}$. Thinking in terms of the quotient graph with respect to the Anstee decomposition, the set A represents one of the edges of the quotient Petersen graph, thus the setwise stabilizer of A in G should have index 15. (In fact, this is true and the whole stablizer is Sylow 2-subgroup of order 32 which is isomorphic to $(\mathbb{Z}_2 \times D_4) : \mathbb{Z}_2$.)

We however need just one permutation g_3 from this stabilizer, where $g_3 = (0, 20, 3, 21, 2, 22, 1, 23)(4, 28, 19, 33, 6, 30, 17, 35)(5, 31, 16, 32, 7, 29, 18, 34)$ $(8, 36, 15, 25, 10, 38, 13, 27)(9, 39, 12, 24, 11, 37, 14, 26)$. It is easy to check that indeed $g_3 \in G$.

Let us now consider group $\langle g_1, g_2, g_3\rangle$. Easy inspection shows that g_3 normalizes $\langle g_1, g_2\rangle$ and transposes two orbits of $\langle g_1, g_2\rangle$ on Ω. Therefore group $Q = \langle g_1, g_2, g_3\rangle$ has order 80 and acts transitively on the set Ω.

Simple geometrical or group theoretical arguments show that $G \cong \mathbb{Z}_4.Y$, where $Y \cong AGL(1, 5)$.

Now we define $R \cong \langle g_1^4, g_3\rangle$. It is evident that R is a transitive group and is a proper subgroup of Q. Thus R is a required regular subgroup of order 40. □

10.3 Coherent closure of the graph $\mathcal{R}$

An intersection diagram of graph $\mathcal{R}$ presented in Figure 10.1, was constructed as follows:

- Fix vertex 0.
- Consider set $N(0) = N_1(0)$ of the neighbors of 0.
- Consider sets $N_2(0)$ and $N_3(0)$ of the vertices at distance 2 and 3 from 0.
- Split set $N_2(0)$ into two sets $N_{2,1}(0)$ and $N_{2,2}(0)$, where $N_{2,2}(0)$ contains those and only those vertices from $N_2(0)$ which have a neighbor in $N_3(0)$.
- Check that the obtained partition is equitable.

The sets in the partition are as follows: $N_0(0) = \{0\}$,
$N_1(0) = \{8, 12, 25, 28, 34, 39\}$,
$N_{2,1}(0) = \{4, 5, 6, 7, 16, 17, 18, 19, 20, 21, 22, 23\}$,
$N_{2,2}(0) = \{9, 10, 11, 13, 14, 15, 24, 26, 27, 29, 30, 31, 32, 33, 35, 36, 37, 38\}$,
$N_3(0) = \{1, 2, 3\}$

Proposition 10.3. *a) Coherent closure $\langle\langle\mathcal{R}\rangle\rangle$ of graph $\mathcal{R}$ is an association scheme with 4 classes and valencies 1, 6, 12, 18, 3.*

b) This association scheme is proper class II Higmanian rank 5 scheme which belongs to type a2.

Proof. a) We use Lemma 2.1 in conjunction with Proposition 10.2 and above observation about the intersection diagram of $\mathcal{R}$. We already know that $Aut(\mathcal{R})$ is transitive, thus $\langle\langle\mathcal{R}\rangle\rangle$ is an association scheme.

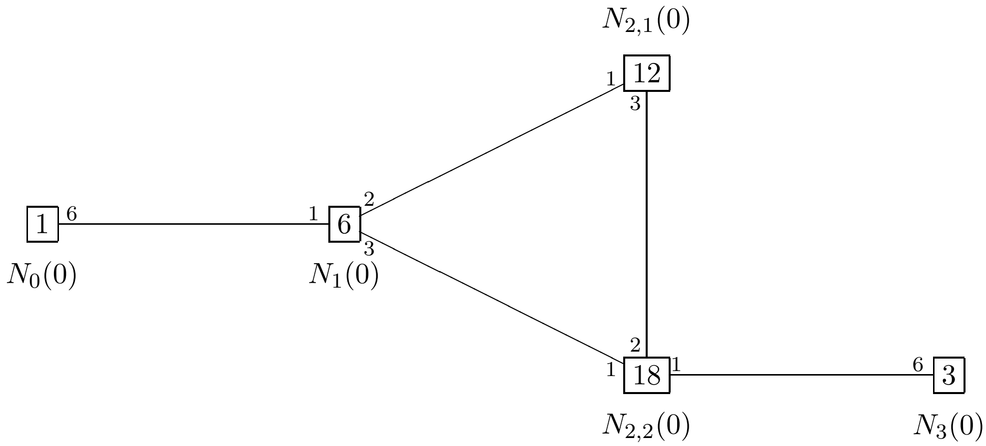

Figure 10.1: Intersection diagram of graph $\mathcal{R}$

b) Note that graph $\mathcal{R}$ is connected of diameter 3. Simple inspection of valencies shows that the distance 3 graph of $\mathcal{R}$ provides, together with reflexive relation the only parabolic in $\langle\langle\mathcal{R}\rangle\rangle$. Note also that the quotient graph of $\mathcal{R}$ is $\overline{P}$.

$\square$

We use notation $\mathfrak{m}_{a2}$ for the Higmanian association scheme $\langle\langle\mathcal{R}\rangle\rangle$ with valencies 1, 3, 6, 18, 12 (according to the ordering used in Proposition 5.2). To the best of our knowledge this is first example of scheme of type a2.

Theorem 10.4. *Let $\mathfrak{m}'$ be an association scheme algebraically isomorphic to $\mathfrak{m}_{a2}$. Then $\mathfrak{m}'$ is combinatorially isomorphic to $\mathfrak{m}_{a2}$. (In other words, $\mathfrak{m}_{a2}$ is uniquely determined up to isomorphism by its tensor of intersection numbers.)*

Proof. Denote by $A_0 = I, A_1, A_2 = S, A_3, A_4$ basic matrices of scheme $\mathfrak{m}'$. We use the same labeling of matrices as in Proposition 5.2. S is the Anstee notation for adjacency matrix of Anstee graph.

According to the formal definition of the Anstee graph we get

$$A_2^2 = 6I + A_3 + A_4.$$

The same formula for the graph $\mathcal{R}$ and the same interpretation of the Anstee graph is visible from Figure 10.1 (remember that here A_2 represents the adjacency matrix of $\mathcal{R}$). Let us now consider the basic graph $\mathcal{R}'$ defined by the matrix A_2'. Clearly this is a connected regular graph of valency 6. Let us construct for $\mathcal{R}'$ the intersection diagram in the same way as it was done for $\mathcal{R}$. Algebraic isomorphism of $\mathfrak{m}_{a2}$ and $\mathfrak{m}'$ sends $\mathcal{R}$ to $\mathcal{R}'$, therefore for $\mathcal{R}'$ we will get the same intersection diagram.

Reading the diagram we first conclude that $\mathcal{R}'$ does not contain triangles. Also, $\mathcal{R}'$ does not contain quadrangles (because for each vertex y at distance

2 from reference vertex 0, there is exactly one path of length two from y to 0).

Finally in a similar manner we conclude that $\mathcal{R}'$ contains cycles of length 5. Therefore $\mathcal{R}'$ is a regular graph of valency 6 and girth 5, that is cage on 40 vertices. Such cage is unique up to isomorphism. Thus, $\mathfrak{m}_{a2} = \langle\langle \mathcal{R} \rangle\rangle$ and $\mathfrak{m}' = \langle\langle \mathcal{R}' \rangle\rangle$ are isomorphic. $\square$

10.4 Full automorphism group

According to the computations done by COCO, $N = SL(2,5)$ is a subgroup of index 4 in $G = Aut(\mathfrak{m}_{a2})$. (Note that $Aut(\mathcal{R}) = Aut(\mathfrak{m}_{a2})$, because $\langle\langle \mathcal{R} \rangle\rangle = \mathfrak{m}_{a2}$).

Our goal is to "legalize" this knowledge. Using group $SL(2,5)$ and some of its overgroups, we will construct new model of graph $\mathcal{R}$ and will determine its automorphism group.

We consider group $\mathrm{GL}(2,5)$, Let $K = HL(2,5)$ be the subgroup of index 2 in $\mathrm{GL}(2,5)$ consisting of matrices with square determinant (H stands for half). Clearly, $|K| = \frac{1}{2}|\mathrm{GL}(2,5)| = 240$.

Let $V = (\mathbb{G}F(5))^2 \setminus \{0\}$ be the set of non-zero row vectors of $(\mathbb{G}F(5))^2$. It is easy to see that K acts transitively on V by right multiplication of a row by a matrix.

Let $O = \{\{x,y,z\} | x,y,z \in V \wedge x \neq y \wedge x \neq z \wedge y \neq z \wedge x+y+z = 0\}$. We note that 3 elements of a typical subset $\{x,y,z\} \in O$ are pairwise independent, in particular element z is uniquely determined by x, y. Therefore $|O| = \frac{24 \cdot 20}{3!} = 80$.

Now we consider natural action of the group K on the set O by $\{x,y,z\}^A = \{xA, yA, zA\}$. We will regard $o_0 = \{(1,0),(0,1),(4,4)\}$ as a reference point in O.

Proposition 10.5. *Group (K,O) has two orbits, each of length 40.*

Proof. Let K_{o_0} be the stabilizer of o_0 in K. K_{o_0} is of even order because $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in K_{o_0}$, therefore $|K_{o_0}| \neq 3$ implying that (K,O) is intransitive permutation group. Taking into account transitivity of group (K,V), let us distinguish different orbits for elements of O which have form $o = \{(1,0),(a,b),(-1-a,-b)\}$.

Such element o belongs to the orbit of o_0 for every $b \in \{1,4\}$ since the matrix $\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \in K$ maps o_0 to o. Considering "antireference" element $\overline{o_0} = \{(1,0),(0,2),(4,3)\}$ and using matrix $\begin{pmatrix} 1 & 0 \\ 3a & 3b \end{pmatrix} \in K$ we get that o belongs to the orbit of $\overline{o_0}$ if $b \in \{2,3\}$. Note that matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \notin K$

permutes orbits of o_0 and $\overline{o_0}$. Finally, we get just two orbits of equal size, noticing that $(0,1)\begin{pmatrix} 0 & b \\ -1 & -b \end{pmatrix} = (-1,-b)$. $\square$

Let $\Omega = o_0^K$ be the orbit of o_0 under the action of K. Clearly $|\Omega| = 40$.

Proposition 10.6. *Transitive permutation group (K,Ω) has rank 10, with four 2-orbits of valency 1, and six 2-orbits of valency 6.*

Proof. Stabilizer K_{o_0} of o_0 is of order $\frac{240}{40} = 6$. Note that any matrix whose two rows are elements of o_0 is in K_{o_0}, thus we get all of K_{o_0} explicitly:

$$K_{o_0} = \left\{ I, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 1 & 0 \end{pmatrix} \right\}.$$

There are three other elements of Ω stabilized by K_{o_0}, namely $o_1 = \{(2,0),(0,2),(3,3)\}$, $o_2 = \{(3,0),(0,3),(2,2)\}$, $o_3 = \{(4,0),(0,4),(1,1)\}$.

In a similar manner we get 6 representatives of orbits of length 6, thus expiring all of set Ω. Here is a list of 10 2-orbits of (K,Ω) by representatives:

$R_i = (o_0, o_i)^K$ for $i = 0,1,2,3$.
$R_4 = (o_0, \{(0,1),(1,1),(4,3)\})^K$,
$R_5 = (o_0, \{(0,1),(1,2),(4,2)\})^K$,
$R_6 = (o_0, \{(0,2),(2,1),(3,2)\})^K$,
$R_7 = (o_0, \{(0,2),(2,2),(3,1)\})^K$,
$R_8 = (o_0, \{(0,3),(2,1),(3,1)\})^K$,
$R_9 = (o_0, \{(0,4),(1,4),(4,2)\})^K$. $\square$

Proposition 10.7. *Let $\Gamma = (\Omega, R_5)$ be a graph, then Γ is a simple graph of valency 6 and girth 5.*

Proof. R_5 can be described by
$R_5 = \big\{(\{x,y,z\},\{x,2x+y,3x+z\})\big|\{x,y,z\} \in \Omega\big\}$.

Since $(\{x,3x+z,2x+y\},\{x,2x+3x+z = z,3x+3x+y = y\}) \in R_5$ as well, R_5 is symmetric, so Γ is a simple graph. Since $K \leq Aut(\Gamma)$, $Aut(\Gamma)$ is edge transitive, so to show that Γ has no triangles, it is enough to show that there are no triangles containing edge $\{\{(0,1),(1,0),(4,4)\},\{(0,1),(1,2),(4,2)\}\}$.

Neighbors of $o_0 = \{(0,1),(1,0),(4,4)\}$ are $n_1 = \{(0,1),(1,2),(4,2)\}$
$n_2 = \{(0,1),(1,3),(4,1)\}$
$n_3 = \{(1,0),(1,4),(3,1)\}$
$n_4 = \{(1,0),(2,1),(2,4)\}$
$n_5 = \{(2,3),(4,4),(4,3)\}$
$n_6 = \{(4,4),(3,2),(3,4)\}$, and neighbors of n_1 are $\{(0,1),(1,0),(4,4)\}$
$\{(0,1),(1,4),(4,0)\}$
$\{(1,1),(1,2),(3,2)\}$
$\{(1,2),(2,0),(2,3)\}$
$\{(2,2),(4,1),(4,2)\}$

$\{(4,2),(3,0),(3,3)\}$, so they have no common neighbors, and edge $\{o_0, n_1\}$ is not part of a triangle.

Similarly, we list all neighbors of n_2, n_3, n_4, n_5, n_6, and see they have no common neighbors other than o_0, thus Γ has no quadrangles as well.

The matrix $\begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}$ is in K, maps o_1 to n_1 and is of order 5, so we have a cycle of length 5 in Γ, thus Γ is of girth 5. □

Corollary. The graph Γ is isomorphic to the unique cage on 40 vertices, that is to the Anstee-Robertson graph $\mathcal{R}$.

Proposition 10.8. *$G = Aut(\mathcal{R})$ is a group of order 480 which is isomorphic to $\mathbb{Z}_4.S_5$.*

Proof. According to Proposition 10.1, $G \cong \mathbb{Z}_4.Y$, where $Y \leq S_5$. We know from Proposition 10.2 that $Q \leq G$, where $Q \cong \mathbb{Z}_4.AGL(1,5)$. It follows from Proposition 10.5 and 10.6 that $K \leq G$.

Note that $K \cong \mathbb{Z}_4.Y$, where Y is subgroup of index 2 in $PGL(2,5) \cong S_5$. Thus clearly $Y \cong A_5$. Now we get that $G/\mathbb{Z}_4$ is an amalgam of subgroups A_5 and $AGL(1,5)$ which is subgroup of S_5. □

10.5 Locally icosahedral graph on 40 vertices

We have now description of group $G = Aut(\mathcal{R})$ as an amalgam of groups Q of order 80 and K of order 240. Alternative more dogmatic description of G will be $\mathbb{Z}_2.A_5.\mathbb{Z}_2.\mathbb{Z}_2$. In other words, G has a normal subgroup $N \cong SL(2,5)$ with $G/N \cong E_{2^2}$. Therefore, for each of three involutions in E_{2^2} we get a subgroup of index 2 in G. One of these groups is group K considered above, two other are denoted by L, M. To distinguish them, we communicate information obtained with the aid of COCO, namely L is of rank 9 with valencies 1,2,1,6,6,6,6,6,6, while M is of rank 11 with valencies 1,1,2,3,3,3,3,6,6,6,6.

Investigating merging association schemes for $(\Omega, 2 - orb(L,\Omega))$ and $(\Omega, 2 - orb(M,\Omega))$ obtained with the aid of COCO, we see that there are no surprises: besides $(\Omega, 2 - orb(G,\Omega))$ and $\mathfrak{m}$ all other mergings are either Schurian, decomposable or imprimitive rank 3 schemes.

Quite different picture appears from $(\Omega, 2 - orb(K,\Omega))$. Here COCO returns 15 merging schemes. The list of schemes, besides predictable ones, contains also two non-Schurian schemes of rank 9 and 8, the first is obtained via merging of relations R_4 and R_7, while the other via mergings of relations R_6, R_8 and R_1, R_2. The automorphism group of both mergings coincides with group K.

In attempts to explain observed objects we became aware of [BloBBC85].

Proposition 10.9. *a) There exists unique locally icosahedral graph Δ on 40 vertices.*

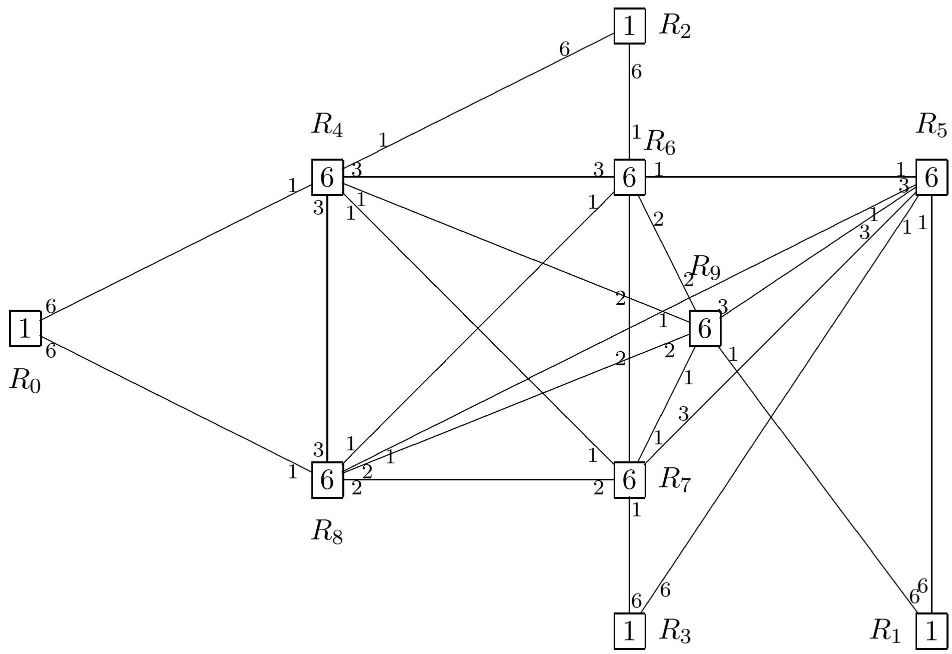

Figure 10.2: Intersection diagram of graph Δ with respect to $Aut(\Delta)$

b) $Aut(\Delta) = K = HL(2,5)$ *is a group of order 240.*

c) The intersection diagram of Δ *with respect to the orbits of a stabilizer of a point in* Δ *is presented in Figure 10.2.*

d) The coherent closure of graph Δ *is a non-Schurian association scheme with 7 classes described above.*

Proof. a) It is proved in [BloBBC85] that there are precisely three locally icosahedral graphs, namely 600-cell on 120 vertices and its quotients on 60 and 40 vertices.

Using GAP and construction (presented in [BloBBC85]) of the unique locally icosahedral graph on 40 vertices, we obtained that its automorphism group is isomorphic (as a permutation group) to (K, Ω). Thus, this graph, Δ, may be described as a merging of 2-orbits of (K, Ω), namely of R_4 and R_9. Using the provided description of the 2-orbits, the reader may easily verify that we indeed get a locally icosahedral graph.

Analysis of mergings of $(\Omega, 2 - orb(G, \Omega))$ shows that $\langle\langle\Delta\rangle\rangle$ is a non-Schurian rank 8 scheme and $Aut(\Delta) = K$. The intersection diagram depicted in Figure 10.2 was constructed with the aid of GAP. It is easy to see that two cells of size 1 (at distance 3 from reference vertex), as well as two cells of size 6 and distance 2 from reference vertex (with external valencies $1^3, 2^2, 3$) may be compressed to a diagram with 8 cells exactly as it is depicted on p. 22 of [BloBBC85]. Now, using this fact of possible compression, together with Lemma 2.1 we get another justification for the existence of rank 8 non-Schurian merging.

□

Remarks.

1. Using COCO-II we found that the color automorphism group of scheme $(\Omega, 2 - orb(K, \Omega))$ has order 480, while the algebraic automorphism group of this scheme is isomorphic to E_{2^2}. In fact, this group (in action on 2-orbits) consists of permutations e, $\tau_1 = (4, 7)$, $\tau_2 = (1, 2)(6, 8)$, $\tau_3 = (4, 7)(1, 2)(6, 8)$.

   Note that τ_3 is induced by the color automorphism group (which is nothing else but group G). The centralizer algebra $V(K, \Omega)$ is commutative, thus existence of τ_2 follows from the well known fact: symmetrization of commutative scheme is an association scheme (see e.g. [BanI84]). In this context the scheme $\langle\langle\Delta\rangle\rangle$ provides a nice illustration of a claim that symmetrization of commutative Schurian scheme is not necessarily Schurian. Finally, τ_2 and τ_3 generate the whole algebraic group. Thus existence of τ_1 and corresponding non Schurian rank 9 merging is a simple by product of all the presented observations.

2. The group K is mentioned in [BloBBC85] as $SL(2, 5) \circ \mathbb{Z}_4$. We believe that we shed some new light on its origin and structure, as well as to links with the Anstee-Robertson graph.

10.6 Some S-rings on 40 points

Our group G of order 480 has two conjugacy classes of regular subgroups of order 40, both classes of size 6. The groups in two classes are isomorphic as permutation groups, and are represented by one of the subgroups, R (as in Proposition 10.2). GAP identifies this group as $\mathbb{Z}_5 : \mathbb{Z}_8$, or group number 3 in catalog of groups of order 40.

G has three subgroups of index 2, $K = HL(2, 5)$, L and M. Of those, only L and M admit the regular group R as a subgroup.

The group R can be defined by generators and relations as follows:

$$\langle x, y | x^5 = y^8 = 1, xy = yx^3\rangle$$

The existence of a regular group R in L and M allows us to interpret the considered association schemes as S-rings over R.

The following proposition is presentation of computer results.

Proposition 10.10. *a) The group L contains a regular subgroup $R \cong \mathbb{Z}_5 : \mathbb{Z}_8$. The transitivity module of L, $\mathcal{T}_1$ has the following basic sets:*
$T_0 = \{e\}$, $T_1 = \{y, xy^2, xy^4, x^4y^6, y^3, x^4\}$, $T_2 = \{y^2, y^6\}$, $T_3 = \{y^4\}$, $T_4 =$

$\{y^5, xy^6, x, x^4y^2, y^7, x^4y^4\}$, $T_5 = \{xy, x^2y^7, x^3y^3, x^2y^4, x^4y^5, x^3y^4\}$, $T_6 = \{xy^3, x^2y^6, x^3y^2, x^2y, x^4y^3, x^3y\}$, $T_7 = \{xy^5, x^2y^3, x^3y^7, x^2, x^4y, x^3\}$, $T_8 = \{xy^7, x^2y^2, x^3y^6, x^2y^5, x^4y^7, x^3y^5\}$.

b) The following S-rings appear as mergings of basic sets of $\mathcal{T}_1$*:*
$\mathfrak{I}_1 = \{T_0, T_1 \cup T_4, T_2, T_3, T_5, T_6 \cup T_8, T_7\}$,
$\mathfrak{I}_2 = \{T_0, T_2 \cup T_3, T_1 \cup T_4, T_5 \cup T_6 \cup T_8, T_7\}$.

c) The group M contains a regular subgroup $R \cong \mathbb{Z}_5 : \mathbb{Z}_8$. The transitivity module of M, $\mathcal{T}_2$, has the following basic sets:
$S_0 = \{e\}$, $S_1 = \{y, y^3, x^4y^4, xy^2, x^4y^6, x\}$, $S_2 = \{y^2, y^6\}$, $S_3 = \{y^4\}$, $S_4 = \{y^5, y^7, x^4, xy^6, x^4y^2, xy^4\}$, $S_5 = \{xy, xy^5, x^2y^7, x^2y^3, x^3y^2, x^3y^6\}$, $S_6 = \{xy^3, x^3, x^2y\}$, $S_7 = \{xy^7, x^3y^4, x^2y^5\}$, $S_8 = \{x^2, x^3y^5, x^4y^7\}$, $S_9 = \{x^2y^2, x^3y^3, x^3y^7, x^4y^5, x^2y^6, x^4y\}$, $S_{10} = \{x^2y^4, x^3y, x^4y^3\}$.

d) The following S-rings appear as mergings of basic sets of $\mathcal{T}_2$*:*
$\mathfrak{I}'_1 = \{S_0, S_1 \cup S_4, S_2, S_3, S_5 \cup S_9, S_6 \cup S_8, S_7 \cup S_{10}\}$,
$\mathfrak{I}'_2 = \{S_0, S_2 \cup S_3, S_1 \cup S_4, S_5 \cup S_9 \cup S_7 \cup S_{10}, S_6 \cup S_8\}$.

e) Robertson graph, $\mathcal{R}$ is a Cayley graph over R with connection set T_7 or $S_6 \cup S_8$.

Proof. $\mathfrak{I}_1$ is the transitivity module coming from G, $\mathfrak{I}_2$ is non-Schurian merging isomorphic to $\mathfrak{m}_{a2}$. Similar explanation for $\mathfrak{I}'_1$, $\mathfrak{I}'_2$. □

11 More objects on 40 and 50 points

In this section we describe a few other coherent configurations and association schemes which are closely related to the main line of presentation in our article.

11.1 Higmanian association scheme of type a3 on 40 points

A Schurian association scheme $\mathfrak{m}_{a3}$ with valencies 1,3,6,6,24 appears twice as a merging of classes in total graph coherent configuration $\mathcal{T}(5)$. Both mergings provide isomorphic scheme with the automorphism group of order 7680. Using GAP, we identify this group as split extension $E_{64} \rtimes S_5$.

The stabilizer of point is a subgroup of order 192, identified by GAP as $D_4 \times S_4$.

Parameter τ for this scheme is equal to 0. We hope to consider this scheme $\mathfrak{m}_{a3}$ with more detail in a forthcoming publication.

11.2 Pentagon coherent configuration on 50 points

Use of coherent configurations as a starting point for construction of interesting combinatorial structures provides more flexibility in comparison with association schemes. In this and next sections we describe two coherent configurations which may serve as a source for construction of HoSi. Roughly speaking, we explain in terms of coherent configurations, how Moore graphs of valency 2 and 3 may be extended to the Moore graph of valency 7.

Taking into account that a possible Moore graph of valency 57 can not have a transitive permutation group ([Asc71], [Cam99]), analysis of the presented objects may help in future more advanced attempts to restrict possible sources from which a Moore graph of valency 57 may appear. We start with a folklore observation, cf. [Haf03].

Proposition 11.1. *The Hoffman-Singleton graph contains 1260 pentagons.*

Proof. Let x be a reference vertex, $N_1(x)$ and $N_2(x)$ sets of vertices on distance 1 and 2 from x. We know from Figure 3.8, that subgraph of HoSi induced by $N_2(x)$ has valency 6. Clearly any edge of this subgraph together with x defines a unique Petersen subgraph in HoSi. Thus for a concrete x we get exactly $\frac{42\cdot 6}{2}$ pentagon subgraphs containing x. Therefore, altogether there are $\frac{126\cdot 50}{5} = 1260$ pentagons in HoSi. □

Let us consider group $D = D_5 \times AGL(1,5)$ of order 200. Our goal is to prove that the stabilizer of an arbitrary pentagon in HoSi coincides with D. For this purpose we consider a certain starting natural intransitive action of D and with its aid define a corresponding coherent configuration with 3 fibers of size 5,25,20. Our initial natural representation of D is on 10 points, $[0,9]$, as follows:

$$D = \langle (0,1,2,3,4),(1,4)(2,3),(5,6,7,8,9)(6,7,9,8)\rangle.$$

We define 3 sets: $\Omega_1 = [0,4]$, Ω_2 is an orbit of 1-factor $\{\{0,5\},\{1,6\},\{2,7\},\{3,8\},\{4,9\}\}$ with respect to D, $\Omega_3 = [0,4]\times[5,9]$. For the reader's convenience the list of elements of the set $\Omega = \Omega_1\cup\Omega_2\cup\Omega_3$ as it was generated by COCO is presented in supplement C. Let $\mathcal{X}_D = (\Omega, 2-orb(D,\Omega))$.

Proposition 11.2. *a)* $\mathcal{X}_D$ *is rank 29 coherent configuration with three fibers* Ω_1, Ω_2 *and* Ω_3 *of type* $\begin{pmatrix}3&1&3\\1&6&3\\3&3&6\end{pmatrix}$.

b) Representatives of 2-orbits of (D,Ω) *are presented in Table 11.1.*

c) $\mathcal{X}_D$ *has 18 mergings association schemes, all the mergings are Schurian.*

0	(0,0)	1	(0,1)	2	(0,2)	3	(0,5)
4	(0,25)	5	(0,26)	6	(0,27)	7	(5,0)
8	(5,5)	9	(5,6)	10	(5,7)	11	(5,10)
12	(5,15)	13	(5,20)	14	(5,25)	15	(5,26)
16	(5,27)	17	(25,0)	18	(25,1)	19	(25,2)
20	(25,5)	21	(25,6)	22	(25,7)	23	(25,25)
24	(25,26)	25	(25,27)	26	(25,30)	27	(25,31)
28	(25,32)						

Table 11.1: 2-orbits of group (D,Ω)

d) Two mergings, namely $(0,8,23)(1,3,5,6,7,9,11,12,13,15,16,18,19,21,22,25,26,27,28)(2,4,10,14,17,20,24)$ *and* $(0,8,23)(1,4,9,14,17,20,25)(2,3,5,6,7,10,11,12,13,15,16,18,19,21,22,24,26,27,28)$ *correspond to the rank 3 association scheme coming from the Hoffman-Singleton graph.*

Proof. Originally, the presented results were obtained with the aid of COCO. Data, which appears in (a) and (b) can be easily justified by hand computations. Results in (c) are indeed computer dependent.

It turns out that (d) may be justified without use of routine computations. For this purpose we need to identify elements of Ω with the vertices in Robertson model of HoSi. First we have to decide whether pentagon invariant with respect to $(D,[0,4])$ or pentagram on the same set will be considered. This explains origin of two possible mergings.

Assume, for example, that we start from the pentagon. After that we need to establish a suitable bijection f between all elements of Ω_2 and 20 vertices of the lower part of the Robertson model (other four pentagons). We also establish a bijection g between elements of Ω_3 and 25 vertices of the upper part of the Robertson model (pentagrams).

As soon as f and g are established, we read our merging and compare it with the rules of adjacency in Robertson model, concluding that they coincide up to the used notation. We leave to the reader this computational exercise. □

Theorem 11.3. *a) All 1260 pentagons of HoSi belong to the same orbit of* $Aut(HoSi)$.

b) The full stabilizer of an arbitrary pentagon in $Aut(HoSi)$ *coincides with group* D.

Proof. It follows from Proposition 11.2 that stabilzier of a certain pentagon in HoSi has order at least 200. Closer analysis of the suggested model in spirit of James [Jam74] and Fan-Schwenk [FanS93] shows that in fact D is the full stabilizer of a selected pentagon. Therefore the orbit of this pentagon in $Aut(HoSi)$ has length 1260. According to Proposition 11.1, this orbit involves all the pentagons in HoSi. □

11.3 Petersen coherent configuration on 50 points

This time we are working in spirit of Jeurissen [Jeu83] and Hafner [Haf03], [Haf04]. We fix arbitrary Petersen subgraph in HoSi, consider its stabilizer in $Aut(HoSi)$, and using corresponding coherent configuration with two fibers of size 10 and 40 recover HoSi.

Proposition 11.4. *The graph HoSi contains exactly 525 copies of the Petersen graph.*

Proof. Simple purely combinatorial arguments (see [Jeu83]) show that:

- HoSi contains exactly 1750 3-claws;
- each 3-claw in HoSi is in precisely 3 Petersen subgraphs;
- Petersen graph has exactly 10 3-claws.

All this information implies that there are precisely $\frac{1750\cdot 3}{10} = 525$ Petersen subgraphs in HoSi. □

Proposition 11.5. *a) $Aut(HoSi)$ acts transitively on the set of Petersen subgraphs.*

b) Stabilizer of a Petersen subgraph in $Aut(HoSi)$ has order 480.

Proof. Implicitely (a) is justified in [Jam74], evident formulation appears in [Haf04]. Now (b) follows from Proposition 11.4. □

Recall that Robertson graph $\mathcal{R}$ is obtained by deletion of P from HoSi. We already know description of $G = Aut(\mathcal{R})$ as a certain group of order 480, while $Aut(P)$ has order 120. We need to get description of the stabilizer of the Petersen graph in $Aut(HoSi)$. For this purpose we consider action of G on the set $\Omega' = \Omega'_1 \cup \Omega'_2$ where Ω'_1 is the vertex set of $\mathcal{R}$, while Ω'_2 is the set of 10 cells in the unique Anstee decomposition of $\mathcal{R}$. We denote by $\mathcal{X}_G$ the coherent cofiguration $(\Omega', 2 - orb(G, \Omega'))$ with two fibers.

Theorem 11.6. *a) $\mathcal{X}_G$ is a rank 16 coherent configuration with two fibers of size 40 and 10 and type $\begin{pmatrix} 7 & 3 \\ 3 & 3 \end{pmatrix}$.*

b) Stabilizer of a cell in Ω'_2 is isomorphic to $D_4 \times S_3$, a group of order 48.

c) Description of 2-orbits is presented in Table 11.2, the list of elements of Ω', as it is produced by GAP, is presented in Supplement C.

0	(0,0)	1	(0,1)	2	(0,2)	3	(0,4)
4	(0,8)	5	(0,11)	6	(0,14)	7	(0,40)
8	(0,41)	9	(0,43)	10	(40,0)	11	(40,2)
12	(40,8)	13	(40,40)	14	(40,41)	15	(40,43)

Table 11.2: Representatives of 2-orbits of group (G, Ω')

d) $\mathcal{X}_G$ *contains unique merging association scheme of rank 3 which corresponds to HoSi.*

Proof. We are aware of the fact that G acts on Ω'_1 as rank 7 group. It is also clear that action of $G \cong \mathbb{Z}_4.S_5$ on Ω'_2 is not faithful, coinciding with the rank 3 action of S_5 on the vertices of the Petersen graph.

Considering vertices on distance 1 and 2 from the vertices of a prescribed cell (which is comprised from nodes of size 1 and 3, depicted in the intersection diagram of graph $\mathcal{R}$), we may elaborate proof of (b). As an immediate conclusion we get orbits of length 4, 12 and 24 of the stabilizer of a selected cell in $\mathcal{R}$ on the vertices in $\mathcal{R}$. This justifies Table 11.2.

To prove (d) we get the only possibility to obtain a regular graph of valency 7 (and its complement), using merging $(0, 13)(5, 7, 10, 14)(1, 2, 3, 4, 6, 8, 9, 11, 12, 16)$.

More routine arguments may allow to prove that the merged graph of valency 7 is strongly regular. □

Corollary. Stabilizer of a Petersen graph in $Aut(HoSi)$ is isomorphic to the group $G = Aut(\mathcal{R})$.

Remark. We attract the reader's attention to two absolutely different functions of the automorphisms of a Petersen graph P inside of HoSi.

If we prescribe a copy of P inside of HoSi then each automorphism of P has four different extensions to an automorphism of HoSi which preserve a selected copy of P.

If we consider $\mathcal{R}$ as a separate graph and select inside of it a copy of P, then only 20 of 120 automorphisms of P may be extended to a suitable automorphism of $\mathcal{R}$.

11.4 Robertson association scheme

We may associate to the Robertson decomposition of HoSi two different subgroups of $Aut(HoSi)$:

- subgroup E which stabilizes each of the 10 cycles of length 5 in the decomposition;
- the normalizer F of E in $Aut(HoSi)$ which permutes 10 cycles of length 5 as a whole.

It turns out that $E \cong \mathbb{Z}_5$, acting semiregularly on the vertices. This immediately implies that $\mathcal{X}_E = (\Omega, 2 - orb(\mathbb{Z}_5, \Omega))$ is a rank 500 group.

In this context group F coincides with the group $CAut(\mathcal{X}_E)$.

The group F turns out to be a transitive permutation group of order 2000. It has a nice description in terms of the stabilizer of an edge in the flag graph of the unique projective plane of order 5 (see [Haf04] for details). Here we provide just the results of computations with the aid of COCO which will be analyzed in forthcoming publication.

Proposition 11.7. *a) Group* (F, Ω) *is 2-closed.*

b) (F, Ω) *is rank 7 group with the subdegrees 1,2,2,5,10,10,20.*

c) The association scheme $(\Omega, 2-orb(F, \Omega))$ *has 8 merging schemes, among them imprimitive rank 5 scheme with valencies 1,4,20,20,5 and two rank 3 schemes corresponding to HoSi.*

Proof. □

11.5 Two association schemes from the master coherent configuration

We again come back to the action of group $G = Aut(\square_5)$ of order 1920 on coherent configuration $\mathfrak{n}$ with 3 fibers of size 40, namely to the consideration of restriction of $\mathfrak{n}$ on second and third fibers.

Proposition 11.8. *a) Both restrictions of* $\mathfrak{n}$ *on second and third fibers define association schemes with 4 classes and valencies 1,3,4,8,24.*

b) These two association schemes are not algebraically isomorphic.

c) Association scheme defined on the third fiber is unique up to isomorphism. It has as one of the merging schemes rank 3 scheme which is defined by the point graph of generalized quadrangle $W(3)$.

Proof. We use COCO in the proof. The uniqueness of a certain rank 5 scheme on 40 points was proved in [BanBB]. In [Ziv2006] it was proved that this Bannai-Bannai-Bannai scheme is isomorphic to the one which appears on the third fiber of $\mathfrak{n}$. □

Remark. In fact, description of the scheme presented in [Ziv2006] is given in slightly different terms in comparison with this article, using as initial object auxiliary graph $\overline{5 \circ K_5}$ instead of $\square_5$. We intend to consider this scheme once more in a forthcoming joint paper of K. Abdukhalikov, E. Bannai, M.K. and M.Z-A.

12 Concluding comments

12.1 Higman's heritage

Almost all concepts considered in this article were originaly discussed by Higman, either evidently or in a hidden form. This counts in particular rank 3 graphs, strongly regular graphs, generalized quadrangles, partial linear spaces, geometric graphs, association schemes, imprimitivity, quotient schemes and many others. While for many lines in Algebraic Combinatorics, Higman should be regarded as one of the founders, it is a concept of a coherent configuration which forever will be attributed to Higman's name.

Association schemes, a particular case of coherent configurations, were introduced by R. C. Bose and his coworkers, its concrete case called translation scheme goes back to I. Schur. In comparison, coherent configurations provide a much more wide natural framework for formulation and classification of various important structures in combinatorics and geometry, like designs, linked designs, partial geometries, etc. In fact, Higman and a few of his close coworkers were the first who started to consider suitable structures in most natural fashion as a coherent configuration with prescribed amount of fibers, prescribed type, and moreover prescribed intersection numbers.

Input of Higman into theory of association schemes is also very significant. Before formulation of CFSG he was the main driving force in classification of rank 3 groups, and one of the founders of the theory of strongly regular graphs. Discovery of Higman-Sims graph on 100 vertices was based on a clever combination of combinatorial and group theoretical arguments. (Note that the graph itself was outlined by Mesner in his Ph.D thesis [Mes56], however lack of the use of group theoretical arguments did not imply earlier discovery of the corresponding sporadic simple group.)

Paper [Hig95] was one of the last Higman's publications. It contains all necessary roots for a start of systematical investigation of rank 5 association schemes. We hope that the presented results of computer algebra experimentation with such objects will help to promote further interest in this topic. In particular, search for all small such schemes seems to be an attractive lead.

12.2 Cellular agebras and graph isomorphism problem

A paper [WeiL68] was published in Russian in a specialized journal with a very limited distribution and was not easily available even in former USSR. It contains a short introduction to the concept of a cellular algebra (almost coinciding with coherent algebra) together with a description of operation "Ы" (part of the title of a famous cultist Soviet fun movie), which is nowadays called Weisfeiler-Leman algorithm for the construction of the coherent closure of a given matrix. The subject and style of the paper, as well as the names of the authors, from the very beginning became a target of attacks of certain antisemitic leaders of Soviet mathematics (see [KliRRT99], [Bab99]). Only in 1976 the results coming from Soviet school became available to a wide mathematical audience (see [Wei76]). After a publication by Higman of his paper [Hig87], the terminology of cellular algebras became obsolete. Nowadays coherent algebras are commonly used as a sign of amalgamation of ideas coming from two independent origins.

The main reason of appearance of cellular algebras was their natural link to the graph isomorphism problem. Indeed, coherent closure $\langle\langle\Gamma\rangle\rangle$ of a given graph Γ may serve as a source of various algebraic invariants of Γ, and this source may be computed in a polynomial time on the number of vertices. While initial naive hopes for an elaboration of efficient algorithm for graph identification quickly failed (see [KliRRT99]), still this concept is very useful in analysis of possible difficulties of the problem.

One more attempt to renew investigations related to this problem was initiated in [Rud02]. Looking at invariants coined in this paper, we came to conclusion that the total graph coherent configuration may provide a more systematic way to attack graph isomorphism problem, see [Ziv], [KliZJ]. Our interest in classification of mergings of total graph coherent configurations stems from the mentioned new incarnation of old hopes to achieve an efficient recognition algorithm.

12.3 Coxeter group $\mathfrak{D}_5$

Coxeter group $\mathfrak{D}_5$ of order 1920 is in a sense one of the main group theoretical heroes of this article.

There are a few natural ways to represent this group in conjunction with a relevant combinatorial or geometrical structure, among them graph $\overline{5 \circ K_5}$ on 10 vertices, $\square_5$ on 16 vertices, 5-dimensional cube on 32 vertices, and last but not least, 40 roots of the corresponding root system. While presentation in [KliZ06] was based on the use of $\overline{5 \circ K_5}$, in this paper we accepted $\square_5$ as the starting structure.

We believe that there is still fresh potential to arrange an alternative new glance on some of considered coherent configurations and association schemes, starting for example from the action on 40 roots. Hopefully this potential will be exploited in subsequent publications.

In a triality (as it is interpreted by Higman in [Hig95]) the starting group G has three non equivalent conjugacy classes of subgroups H_1, H_2, H_3, which are becoming equivalent under action of overgroup $G.S_3$. This implies existence of a coherent configuration with 3 fibers of equal size, as well as of a merging association scheme glueing all the 3 fibers.

Our master coherent configuration $\mathfrak{n}$ serves as a certain weak combinatorial analogue of the classical triality. Indeed, we get here three non-equivalent fibers of the same size, and moreover each fiber induces association scheme with 4 classes. Two of the fibers produce a classical partial linear space $\mathfrak{S}_1$, namely a generalized quadrangle $Q(4,3)$, while other pair of fibers produces partial linear space $\mathfrak{S}_2$ with some nice properties. We refer to [KliR03] where similar exceptional pair of square incidence structures (96 points and 96 lines) was discovered. In both cases a new partial linear space appears as a kind of a satellite of a classical generalized quadrangle.

More such examples are desired.

12.4 Exceptional cages

Moore graphs provide the most exceptional examples of cages. The Hoffman-Singleton graph corresponds to a certain rank 3 group, which appears as a sporadic permutation representation of a "usual" classical group. The existence of such representation probably explains why the graph HoSi was discovered on the early dawn of theory (1960), that is three years before the time when the general concept of a strongly regular graph was coined by Bose in his seminal paper [Bos63].

As it was mentioned, a possible Moore graph of valency 57 can not have a transitive automorphism group, it was already Aschbacher who proved that association scheme with two classes defined by such a graph is non-Schurian. These two facts together imply that the desired association scheme (if it exists and appears as a merging) may come only as a fusion of a suitable coherent configuration with at least two fibers. In this context a few coherent configurations on 50 points exposed in Section 11 may provide a helpful training place in order to guess a good candidate to achieve a Moore graph on 3250 vertices.

Speculating again about possible ways to attempt a Moore graph Γ of valency 57, one may claim that start from a suitable coherent configuration means in a sense that we are taking only part of information about symmetry of Γ. In contrary, another attractive way would be to embed Γ into a suitable larger structure Δ, such that finally $Aut(\Gamma)$ will be a subgroup (possibly proper) of $Aut(\Delta)$.

The approach proposed by us, of total graph coherent configurations in case when $\Gamma = HoSi$, allows to reach a much larger graph Δ such that $Aut(\Delta)$ is isomorphic to the automorphism group of the Higman-Sims graph.

Graph Δ is achieved by a very dogmatic procedure (any insight is not requested), using a computer, as a merging in total graph coherent configuration for Γ. We refer to [KliZJ] for a discussion of similar expectations related to Γ in a role of Moore graph of valency 57.

Note that HoSi may be also regarded as a (non-bipartite) coherent cage. The (bipartite) Levi graphs of generalized polygons provide infinite series of other coherent cages. In this relation the Anstee-Robertson graph is one of the first (if not the first) of sporadic coherent cages. In our eyes, the search for new similar examples may be an attractive lead on the edge between extremal graph theory and coherent configurations.

12.5 Computer and mathematician: a bilateral interplay

Most of results of Higman in area of graph theory and combinatorics were motivated by his expertise from group theory. Though Higman was one of the pioneers in defining efficient techniques which help to distinguish Schurian and non-Schurian association schemes (see e.g. [HesH71]), his main interests were related to the consideration of orbital graphs (as they appear in seminal papers [Hig67], [Sim67]), or 2-orbits in Wielandt's terminology (which is used in this paper).

The concept of orbital graph and more generally of a Schurian coherent configuration may be adequately represented in terms of a group, certain subgroups, and suitable systems of cosets and double cosets. This is the language that was mostly used in papers of Higman, his coworkers and his followers.

Nowadays, with development of computer possibilities, more and more investigation in algebraic combinatorics are related to non-Schurian objects. This dictates certain objective changes in standards of behavior of a mathematician. In many cases an expert trusts more a computer rather than a mathematician. Many computer aided constructions are in principle not observable in a friendly form by a human. In the course of manipulation with coherent configuration of a huge rank, the use of traditional language of coset diagrams may be less efficient.

In this article we are trying to submit a few new patterns of a computer practice with combinatorial objects, which conceptually go back to the methodology of representation of 2-orbits in COCO. We believe that in certain cases these patterns are advantageous, allowing to involve visually clear geometrical images as an auxiliary tool, suitable even for proofs.

12.6 Looking forward

Use of groups of algebraic automorphisms of coherent configurations conceptually goes back to Higman as well as to Weisfeiler and Leman (see [KliMPWZ07] for more information).

Practical manipulations with these objects are heavily computer dependent, this is why interesting non-trivial examples of non-Schurian algebraic mergings and of twins were detected quite recently, [KliMRR05] is one of the first sources. Everything here is in a very beginning. Our empirical observations show that the size of the algebraic group is not a crucial issue. In this article, the reader meets examples of configurations with huge algebraic groups without proper algebraic automorphisms at all and with small groups containing proper automorphisms.

In this context, WFDF configurations are mostly attractive with their large agebraic groups, mostly consisting of proper automorphisms. Our experience to deal with WFDF configurations on 16, 28 and 40 vertices shows that we are quickly approaching upper bound for the opportunities to investigate lattice of subgroups and corresponding algebraic mergings with the aid of modern computer algebra facilities. New theoretical insights are badly needed in order to promote efficient use of this concept.

On other hand, capacity of possibilities of WFDF configurations seems to be unlimited. Recent discovery of numerous new infinite series of strongly regular graphs in [Muz07] confirms such hope.

We also mention other suggested innovations: proof of the uniqueness of association scheme $\mathfrak{m}_{a2}$ and computer enumeration of all Deza families in Higmanian houses on 40 points. The latter problem still has reserve for a more clever solution: to avoid knowledge of Ted Spence's catalog of strongly regular graphs, approaching only "promising" graphs, leading to at least one solution.

This text is regarded by the authors as a comprehensive technical report about the results of the fulfilled project. An essentially reduced version will be soon submitted for a regular publication.

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Supplements

A Data related to partial linear space $\mathfrak{S}_2$

40 quadrangles of $\square_5$

0	0,1,2,3	1	0,1,4,5	2	0,1,8,9	3	0,1,14,15
4	0,2,4,6	5	0,2,8,10	6	0,2,13,15	7	0,4,8,12
8	0,4,11,15	9	0,7,8,15	10	1,3,5,7	11	1,3,9,11
12	1,3,12,14	13	1,5,9,13	14	1,5,10,14	15	1,6,9,14
16	2,3,6,7	17	2,3,10,11	18	2,3,12,13	19	2,5,10,13
20	2,6,9,13	21	2,6,10,14	22	3,4,11,12	23	3,7,8,12
24	3,7,11,15	25	4,5,6,7	26	4,5,10,11	27	4,5,12,13
28	4,6,9,11	29	4,6,12,14	30	5,7,8,10	31	5,7,13,15
32	6,7,8,9	33	6,7,14,15	34	8,9,10,11	35	8,9,12,13
36	8,10,12,14	37	9,11,13,15	38	10,11,14,15	39	12,13,14,15

40 edges of $\square_5$ and lines of $\mathfrak{S}_2$

0	0,1	0,1,2,3	1	0,2	0,4,5,6	2	0,4	1,4,7,8	3	0,8	2,5,7,9
4	0,15	3,6,8,9	5	1,3	0,10,11,12	6	1,5	1,10,13,14	7	1,9	2,11,13,15
8	1,14	3,12,14,15	9	2,3	0,16,17,18	10	2,6	4,16,20,21	11	2,10	5,17,19,21
12	2,13	6,18,19,20	13	3,7	10,16,23,24	14	3,11	11,17,22,24	15	3,12	12,18,22,23
16	4,5	1,25,26,27	17	4,6	4,25,28,29	18	4,11	8,22,26,28	19	4,12	7,22,27,29
20	5,7	10,25,30,31	21	5,10	14,19,26,30	22	5,13	13,19,27,31	23	6,7	16,25,32,33
24	6,9	15,20,28,32	25	6,14	15,21,29,33	26	7,8	9,23,30,32	27	7,15	9,24,31,33
28	8,9	2,32,34,35	29	8,10	5,30,34,36	30	8,12	7,23,35,36	31	9,11	11,28,34,37
32	9,13	13,20,35,37	33	10,11	17,26,34,38	34	10,14	14,21,36,38	35	11,15	8,24,37,38
36	12,13	18,27,35,39	37	12,14	12,29,36,39	38	13,15	6,31,37,39	39	14,15	3,33,38,39

14 paths of length 2 in Γ_2

$\{\{0,1,2,3\},\{0,1,8,9\},\{0,1,4,5\}\}$
$\{\{0,1,2,3\},\{0,1,14,15\},\{0,1,4,5\}\}$
$\{\{0,1,2,3\},\{0,2,4,6\},\{0,1,4,5\}\}$
$\{\{0,1,2,3\},\{1,3,5,7\},\{0,1,4,5\}\}$
$\{\{0,1,2,3\},\{0,1,4,5\},\{0,4,8,12\}\}$
$\{\{0,1,2,3\},\{0,1,8,9\},\{0,4,8,12\}\}$
$\{\{0,1,2,3\},\{0,2,4,6\},\{0,4,8,12\}\}$
$\{\{0,1,2,3\},\{0,2,8,10\},\{0,4,8,12\}\}$
$\{\{0,1,2,3\},\{0,1,4,5\},\{4,5,6,7\}\}$
$\{\{0,1,2,3\},\{0,2,4,6\},\{4,5,6,7\}\}$
$\{\{0,1,2,3\},\{1,3,5,7\},\{4,5,6,7\}\}$
$\{\{0,1,2,3\},\{2,3,6,7\},\{4,5,6,7\}\}$
$\{\{0,1,2,3\},\{0,1,4,5\},\{4,5,10,11\}\}$
$\{\{0,1,2,3\},\{2,3,10,11\},\{4,5,10,11\}\}$

B Data related to algebraic group of WFDF coherent configuration on 40 points

Representatives of basic relations of W

0	$\langle 0,0\rangle$	1	$\langle 0,1\rangle$	2	$\langle 0,2\rangle$	3	$\langle 0,3\rangle$	4	$\langle 0,4\rangle$
5	$\langle 0,5\rangle$	6	$\langle 0,6\rangle$	7	$\langle 0,7\rangle$	8	$\langle 0,8\rangle$	9	$\langle 0,9\rangle$
10	$\langle 0,25\rangle$	11	$\langle 0,26\rangle$	12	$\langle 0,27\rangle$	13	$\langle 0,28\rangle$	14	$\langle 0,29\rangle$
15	$\langle 0,34\rangle$	16	$\langle 0,35\rangle$	17	$\langle 0,36\rangle$	18	$\langle 0,39\rangle$	19	$\langle 1,0\rangle$
20	$\langle 1,1\rangle$	21	$\langle 1,2\rangle$	22	$\langle 1,3\rangle$	23	$\langle 1,4\rangle$	24	$\langle 1,5\rangle$
25	$\langle 1,6\rangle$	26	$\langle 1,7\rangle$	27	$\langle 1,8\rangle$	28	$\langle 1,9\rangle$	29	$\langle 1,16\rangle$
30	$\langle 1,17\rangle$	31	$\langle 1,18\rangle$	32	$\langle 1,20\rangle$	33	$\langle 1,21\rangle$	34	$\langle 1,34\rangle$
35	$\langle 1,35\rangle$	36	$\langle 1,36\rangle$	37	$\langle 1,38\rangle$	38	$\langle 2,0\rangle$	39	$\langle 2,1\rangle$
40	$\langle 2,2\rangle$	41	$\langle 2,3\rangle$	42	$\langle 2,4\rangle$	43	$\langle 2,5\rangle$	44	$\langle 2,6\rangle$
45	$\langle 2,7\rangle$	46	$\langle 2,8\rangle$	47	$\langle 2,9\rangle$	48	$\langle 2,16\rangle$	49	$\langle 2,17\rangle$
50	$\langle 2,18\rangle$	51	$\langle 2,19\rangle$	52	$\langle 2,21\rangle$	53	$\langle 2,25\rangle$	54	$\langle 2,27\rangle$
55	$\langle 2,29\rangle$	56	$\langle 2,33\rangle$	57	$\langle 3,0\rangle$	58	$\langle 3,1\rangle$	59	$\langle 3,2\rangle$
60	$\langle 3,3\rangle$	61	$\langle 3,4\rangle$	62	$\langle 3,5\rangle$	63	$\langle 3,6\rangle$	64	$\langle 3,7\rangle$
65	$\langle 3,8\rangle$	66	$\langle 3,9\rangle$	67	$\langle 3,16\rangle$	68	$\langle 3,17\rangle$	69	$\langle 3,18\rangle$
70	$\langle 3,19\rangle$	71	$\langle 3,20\rangle$	72	$\langle 3,25\rangle$	73	$\langle 3,26\rangle$	74	$\langle 3,28\rangle$
75	$\langle 3,32\rangle$	76	$\langle 4,0\rangle$	77	$\langle 4,1\rangle$	78	$\langle 4,2\rangle$	79	$\langle 4,3\rangle$
80	$\langle 4,4\rangle$	81	$\langle 4,5\rangle$	82	$\langle 4,6\rangle$	83	$\langle 4,7\rangle$	84	$\langle 4,8\rangle$
85	$\langle 4,9\rangle$	86	$\langle 4,10\rangle$	87	$\langle 4,11\rangle$	88	$\langle 4,12\rangle$	89	$\langle 4,13\rangle$
90	$\langle 4,14\rangle$	91	$\langle 4,34\rangle$	92	$\langle 4,35\rangle$	93	$\langle 4,36\rangle$	94	$\langle 4,37\rangle$
95	$\langle 5,0\rangle$	96	$\langle 5,1\rangle$	97	$\langle 5,2\rangle$	98	$\langle 5,3\rangle$	99	$\langle 5,4\rangle$
100	$\langle 5,5\rangle$	101	$\langle 5,6\rangle$	102	$\langle 5,7\rangle$	103	$\langle 5,8\rangle$	104	$\langle 5,9\rangle$
105	$\langle 5,10\rangle$	106	$\langle 5,11\rangle$	107	$\langle 5,12\rangle$	108	$\langle 5,13\rangle$	109	$\langle 5,15\rangle$
110	$\langle 5,25\rangle$	111	$\langle 5,27\rangle$	112	$\langle 5,29\rangle$	113	$\langle 5,31\rangle$	114	$\langle 6,0\rangle$
115	$\langle 6,1\rangle$	116	$\langle 6,2\rangle$	117	$\langle 6,3\rangle$	118	$\langle 6,4\rangle$	119	$\langle 6,5\rangle$
120	$\langle 6,6\rangle$	121	$\langle 6,7\rangle$	122	$\langle 6,8\rangle$	123	$\langle 6,9\rangle$	124	$\langle 6,10\rangle$
125	$\langle 6,11\rangle$	126	$\langle 6,12\rangle$	127	$\langle 6,14\rangle$	128	$\langle 6,15\rangle$	129	$\langle 6,25\rangle$
130	$\langle 6,26\rangle$	131	$\langle 6,28\rangle$	132	$\langle 6,30\rangle$	133	$\langle 7,0\rangle$	134	$\langle 7,1\rangle$
135	$\langle 7,2\rangle$	136	$\langle 7,3\rangle$	137	$\langle 7,4\rangle$	138	$\langle 7,5\rangle$	139	$\langle 7,6\rangle$
140	$\langle 7,7\rangle$	141	$\langle 7,8\rangle$	142	$\langle 7,9\rangle$	143	$\langle 7,10\rangle$	144	$\langle 7,11\rangle$
145	$\langle 7,13\rangle$	146	$\langle 7,14\rangle$	147	$\langle 7,15\rangle$	148	$\langle 7,16\rangle$	149	$\langle 7,17\rangle$
150	$\langle 7,21\rangle$	151	$\langle 7,24\rangle$	152	$\langle 8,0\rangle$	153	$\langle 8,1\rangle$	154	$\langle 8,2\rangle$
155	$\langle 8,3\rangle$	156	$\langle 8,4\rangle$	157	$\langle 8,5\rangle$	158	$\langle 8,6\rangle$	159	$\langle 8,7\rangle$
160	$\langle 8,8\rangle$	161	$\langle 8,9\rangle$	162	$\langle 8,10\rangle$	163	$\langle 8,12\rangle$	164	$\langle 8,13\rangle$
165	$\langle 8,14\rangle$	166	$\langle 8,15\rangle$	167	$\langle 8,16\rangle$	168	$\langle 8,18\rangle$	169	$\langle 8,20\rangle$
170	$\langle 8,23\rangle$	171	$\langle 9,0\rangle$	172	$\langle 9,1\rangle$	173	$\langle 9,2\rangle$	174	$\langle 9,3\rangle$
175	$\langle 9,4\rangle$	176	$\langle 9,5\rangle$	177	$\langle 9,6\rangle$	178	$\langle 9,7\rangle$	179	$\langle 9,8\rangle$
180	$\langle 9,9\rangle$	181	$\langle 9,11\rangle$	182	$\langle 9,12\rangle$	183	$\langle 9,13\rangle$	184	$\langle 9,14\rangle$
185	$\langle 9,15\rangle$	186	$\langle 9,17\rangle$	187	$\langle 9,18\rangle$	188	$\langle 9,19\rangle$	189	$\langle 9,22\rangle$

Generators of E_{2^4} in action on Ω

$$
\begin{gathered}
(4,10)(5,11)(6,12)(7,13)(8,14)(9,15)(19,22)(20,23)(21,24)(28,30)(29,31)(36,37),\\
(1,16)(2,17)(3,18)(7,21)(8,20)(9,19)(13,24)(14,23)(15,22)(26,32)(27,33)(35,38),\\
(0,25)(2,27)(3,26)(5,29)(6,28)(9,22)(11,31)(12,30)(15,19)(17,33)(18,32)(34,39),\\
(0,34)(1,35)(3,32)(4,36)(6,30)(8,23)(10,37)(12,28)(14,20)(16,38)(18,26)(25,39)
\end{gathered}
$$

List of Fibers of W

$$
\begin{array}{ccccc}
\{0,25,34,39\} & \{1,16,35,38\} & \{2,17,27,33\} & \{3,18,26,32\} & \{4,10,36,37\}\\
\{5,11,29,31\} & \{6,12,28,30\} & \{7,13,21,24\} & \{8,14,20,23\} & \{9,15,19,22\}
\end{array}
$$

Generators of $CAut(W)$ in action on Ω

$$
\begin{gathered}
(8,14)(20,23)\\
(2,3)(5,6)(7,8)(11,12)(13,14)(17,18)(20,21)(23,24)(26,27)(28,29)(30,31)(32,33)(34,39)(35,38)(36,37)\\
(1,2)(4,5)(8,9)(10,11)(14,15)(16,17)(19,20)(22,23)(25,34)(26,32)(27,35)(28,30)(29,36)(31,37)(33,38)\\
(1,4)(2,5)(3,6)(10,16)(11,17)(12,18)(13,21)(14,20)(15,19)(26,28)(27,29)(30,32)(31,33)(35,36)(37,38)\\
(0,1)(5,7)(6,8)(11,13)(12,14)(16,25)(17,27)(18,26)(19,22)(20,28)(21,29)(23,30)(24,31)(34,35)(38,39)
\end{gathered}
$$

Generators of $CAut(W)$ in action on R

$$
(84,90)(122,127)(141,146)(156,162)(158,163)(159,164)(161,166)(179,184)
$$

$$
\begin{gathered}
(2,3)(5,6)(7,8)(11,12)(13,14)(15,18)(21,22)(24,25)(26,27)(30,31)(32,33)\\
(35,37)(38,57)(39,58)(40,60)(41,59)(42,61)(43,63)(44,62)(45,65)(46,64)(47,66)\\
(48,67)(49,69)(50,68)(51,70)(52,71)(53,72)(54,73)(55,74)(56,75)(78,79)(81,82)\\
(83,84)(87,88)(89,90)(93,94)(95,114)(96,115)(97,117)(98,116)(99,118)(100,120)\\
(101,119)(102,122)(103,121)(104,123)(105,124)(106,126)(107,125)(108,127)(109,128)\\
(110,129)(111,130)(112,131)(113,132)(133,152)(134,153)(135,155)(136,154)(137,156)\\
(138,158)(139,157)(140,160)(141,159)(142,161)(143,162)(144,163)(145,165)(146,164)\\
(147,166)(148,167)(149,168)(150,169)(151,170)(173,174)(176,177)(178,179)(181,182)\\
(183,184)(186,187)
\end{gathered}
$$

$$
\begin{gathered}
(1,2)(4,5)(8,9)(10,15)(12,16)(14,17)(19,38)(20,40)(21,39)(22,41)(23,43)\\
(24,42)(25,44)(26,45)(27,47)(28,46)(29,49)(30,48)(31,50)(32,51)(33,52)(34,53)\\
(35,54)(36,55)(37,56)(58,59)(61,62)(65,66)(67,68)(70,71)(73,75)(76,95)(77,97)\\
(78,96)(79,98)(80,100)(81,99)(82,101)(83,102)(84,104)(85,103)(86,106)\\
(87,105)(88,107)(89,108)(90,109)(91,110)(92,111)(93,112)(94,113)(115,116)\\
(118,119)(122,123)(124,125)(127,128)(131,132)(134,135)(137,138)(141,142)(143,144)\\
(146,147)(148,149)(152,171)(153,173)(154,172)(155,174)(156,176)(157,175)(158,177)\\
(159,178)(160,180)(161,179)(162,181)(163,182)(164,183)(165,185)(166,184)(167,186)\\
(168,187)(169,188)(170,189)
\end{gathered}
$$

$$
\begin{gathered}
(1,4)(2,5)(3,6)(11,13)(12,14)(16,17)(19,76)(20,80)(21,81)(22,82)(23,77)\\
(24,78)(25,79)(26,83)(27,84)(28,85)(29,86)(30,87)(31,88)(32,90)(33,89)(34,91)\\
(35,93)(36,92)(37,94)(38,95)(39,99)(40,100)(41,101)(42,96)(43,97)(44,98)\\
(45,102)(46,103)(47,104)(48,105)(49,106)(50,107)(51,109)(52,108)(53,110)\\
(54,112)(55,111)(56,113)(57,114)(58,118)(59,119)(60,120)(61,115)(62,116)\\
(63,117)(64,121)(65,122)(66,123)(67,124)(68,125)(69,126)(70,128)(71,127)\\
(72,129)(73,131)(74,130)(75,132)(134,137)(135,138)(136,139)(143,148)(144,149)\\
(145,150)(153,156)(154,157)(155,158)(162,167)(163,168)(165,169)(172,175)(173,176)\\
(174,177)(181,186)(182,187)(185,188)
\end{gathered}
$$

$$
\begin{gathered}
(0,20)(1,19)(2,21)(3,22)(4,23)(5,26)(6,27)(7,24)(8,25)(9,28)(10,29)\\
(11,31)(12,30)(13,32)(14,33)(15,35)(16,34)(17,36)(18,37)(38,39)(43,45)(44,46)\\
(48,53)(49,54)(52,55)(57,58)(62,64)(63,65)(67,72)(69,73)(71,74)(76,77)(81,83)\\
(82,84)(87,89)(88,90)(91,92)(95,134)(96,133)(97,135)(98,136)(99,137)(100,140)\\
(101,141)(102,138)(103,139)(104,142)(105,143)(106,145)(107,146)(108,144)(109,147)\\
(110,148)(111,149)(112,150)(113,151)(114,153)(115,152)(116,154)(117,155)(118,156)\\
(119,159)(120,160)(121,157)(122,158)(123,161)(124,162)(125,164)(126,165)(127,163)\\
(128,166)(129,167)(130,168)(131,169)(132,170)(171,172)(176,178)(177,179)(181,183)\\
(182,184)(188,189)
\end{gathered}
$$

Generators of $AAut(W)$ in action on R

(41, 50)(47, 51)(59, 68)(173, 186)

(2, 3)(5, 6)(7, 8)(11, 12)(13, 14)(15, 18)(21, 22)(24, 25)(26, 27)(30, 31)(32, 33)
(35, 37)(38, 57)(39, 58)(40, 60)(41, 59)(42, 61)(43, 63)(44, 62)(45, 65)(46, 64)(47, 66)
(48, 67)(49, 69)(50, 68)(51, 70)(52, 71)(53, 72)(54, 73)(55, 74)(56, 75)(78, 79)(81, 82)
(83, 84)(87, 88)(89, 90)(93, 94)(95, 114)(96, 115)(97, 117)(98, 116)(99, 118)(100, 120)
(101, 119)(102, 122)(103, 121)(104, 123)(105, 124)(106, 126)(107, 125)(108, 127)(109, 128)
(110, 129)(111, 130)(112, 131)(113, 132)(133, 152)(134, 153)(135, 155)(136, 154)(137, 156)
(138, 158)(139, 157)(140, 160)(141, 159)(142, 161)(143, 162)(144, 163)(145, 165)(146, 164)
(147, 166)(148, 167)(149, 168)(150, 169)(151, 170)(173, 174)(176, 177)(178, 179)(181, 182)
(183, 184)(186, 187)

(1, 2)(4, 5)(8, 9)(10, 15)(12, 16)(14, 17)(19, 38)(20, 40)(21, 39)(22, 41)(23, 43)
(24, 42)(25, 44)(26, 45)(27, 47)(28, 46)(29, 49)(30, 48)(31, 50)(32, 51)(33, 52)(34, 53)
(35, 54)(36, 55)(37, 56)(58, 59)(61, 62)(65, 66)(67, 68)(70, 71)(73, 75)(76, 95)(77, 97)
(78, 96)(79, 98)(80, 100)(81, 99)(82, 101)(83, 102)(84, 104)(85, 103)(86, 106)
(87, 105)(88, 107)(89, 108)(90, 109)(91, 110)(92, 111)(93, 112)(94, 113)(115, 116)
(118, 119)(122, 123)(124, 125)(127, 128)(131, 132)(134, 135)(137, 138)(141, 142)(143, 144)
(146, 147)(148, 149)(152, 171)(153, 173)(154, 172)(155, 174)(156, 176)(157, 175)(158, 177)
(159, 178)(160, 180)(161, 179)(162, 181)(163, 182)(164, 183)(165, 185)(166, 184)(167, 186)
(168, 187)(169, 188)(170, 189)

(1, 4)(2, 5)(3, 6)(11, 13)(12, 14)(16, 17)(19, 76)(20, 80)(21, 81)(22, 82)(23, 77)
(24, 78)(25, 79)(26, 83)(27, 84)(28, 85)(29, 86)(30, 87)(31, 88)(32, 90)(33, 89)(34, 91)
(35, 93)(36, 92)(37, 94)(38, 95)(39, 99)(40, 100)(41, 101)(42, 96)(43, 97)(44, 98)
(45, 102)(46, 103)(47, 104)(48, 105)(49, 106)(50, 107)(51, 109)(52, 108)(53, 110)
(54, 112)(55, 111)(56, 113)(57, 114)(58, 118)(59, 119)(60, 120)(61, 115)(62, 116)
(63, 117)(64, 121)(65, 122)(66, 123)(67, 124)(68, 125)(69, 126)(70, 128)(71, 127)
(72, 129)(73, 131)(74, 130)(75, 132)(134, 137)(135, 138)(136, 139)(143, 148)(144, 149)
(145, 150)(153, 156)(154, 157)(155, 158)(162, 167)(163, 168)(165, 169)(172, 175)(173, 176)
(174, 177)(181, 186)(182, 187)(185, 188)

(0, 20)(1, 19)(2, 21)(3, 22)(4, 23)(5, 26)(6, 27)(7, 24)(8, 25)(9, 28)(10, 29)
(11, 31)(12, 30)(13, 32)(14, 33)(15, 35)(16, 34)(17, 36)(18, 37)(38, 39)(43, 45)(44, 46)
(48, 53)(49, 54)(52, 55)(57, 58)(62, 64)(63, 65)(67, 72)(69, 73)(71, 74)(76, 77)(81, 83)
(82, 84)(87, 89)(88, 90)(91, 92)(95, 134)(96, 133)(97, 135)(98, 136)(99, 137)(100, 140)
(101, 141)(102, 138)(103, 139)(104, 142)(105, 143)(106, 145)(107, 146)(108, 144)(109, 147)
(110, 148)(111, 149)(112, 150)(113, 151)(114, 153)(115, 152)(116, 154)(117, 155)(118, 156)
(119, 159)(120, 160)(121, 157)(122, 158)(123, 161)(124, 162)(125, 164)(126, 165)(127, 163)
(128, 166)(129, 167)(130, 168)(131, 169)(132, 170)(171, 172)(176, 178)(177, 179)(181, 183)
(182, 184)(188, 189)

Merging of classes of W resulting in scheme isomorphic to $\mathfrak{m}_{6.1}$

0, 20, 40, 60, 80, 100, 120, 140, 160, 180
1, 2, 3, 4, 5, 6, 19, 21, 22, 23, 26, 27, 38, 39, 41, 43, 45, 47, 57, 58, 59, 63, 65, 66, 76, 77, 81, 82, 83, 84, 95, 97, 99, 101, 102, 104, 114, 117, 118, 119, 122, 123, 134, 135, 137, 138, 141, 142, 153, 155, 156, 158, 159, 161, 173, 174, 176, 177, 178, 179
7, 8, 9, 24, 25, 28, 42, 44, 46, 61, 62, 64, 78, 79, 85, 96, 98, 103, 115, 116, 121, 133, 136, 139, 152, 154, 157, 171, 172, 175
10, 15, 18, 29, 35, 37, 49, 54, 56, 69, 73, 75, 86, 93, 94, 106, 112, 113, 126, 131, 132, 145, 150, 151, 165, 169, 170, 185, 188, 189
11, 12, 13, 14, 16, 17, 30, 31, 32, 33, 34, 36, 48, 50, 51, 52, 53, 55, 67, 68, 70, 71, 72, 74, 87, 88, 89, 90, 91, 92, 105, 107, 108, 109, 110, 111, 124, 125, 127, 128, 129, 130, 143, 144, 146, 147, 148, 149, 162, 163, 164, 166, 167, 168, 181, 182, 183, 184, 186, 187

Merging of classes of W resulting in scheme isomorphic to $\mathfrak{m}_{2.1}$

0, 20, 40, 60, 80, 100, 120, 140, 160, 180
1, 3, 5, 12, 13, 17, 19, 21, 27, 31, 33, 36, 39, 50, 51, 52, 53, 55, 57, 66, 67, 68, 71, 74, 81, 82, 83, 90, 91, 92, 95, 99, 102, 104, 107, 111, 118, 125, 127, 128, 129, 130, 137, 138, 141, 147, 148, 149, 153, 159, 162, 163, 166, 168, 174, 176, 182, 183, 184, 186
2, 4, 6, 11, 14, 16, 22, 23, 26, 30, 32, 34, 38, 41, 43, 45, 47, 48, 58, 59, 63, 65, 70, 72, 76, 77, 84, 87, 88, 89, 97, 101, 105, 108, 109, 110, 114, 117, 119, 122, 123, 124, 134, 135, 142, 143, 144, 146, 155, 156, 158, 161, 164, 167, 173, 177, 178, 179, 181, 187
7, 8, 9, 24, 25, 28, 42, 44, 46, 61, 62, 64, 78, 79, 85, 96, 98, 103, 115, 116, 121, 133, 136, 139, 152, 154, 157, 171, 172, 175
[10, 15, 18, 29, 35, 37, 49, 54, 56, 69, 73, 75, 86, 93, 94, 106, 112, 113, 126, 131, 132, 145, 150, 151, 165, 169, 170, 185, 188, 189

Merging of classes of W resulting in scheme isomorphic to $\mathfrak{m}_{7.1}$

0, 20, 40, 60, 80, 100, 120, 140, 160, 180
1, 3, 4, 12, 13, 14, 19, 21, 31, 32, 33, 36, 39, 41, 45, 51, 53, 55, 57, 59, 63, 66, 67, 71, 76, 82, 83, 84, 87, 92, 101, 102, 105, 109, 110, 111, 117, 118, 119, 123, 127, 129, 135, 137, 138, 141, 147, 148, 156, 159, 161, 163, 167, 168, 174, 177, 179, 181, 183, 186
2, 5, 6, 11, 16, 17, 22, 23, 26, 27, 30, 34, 38, 43, 47, 48, 50, 52, 58, 65, 68, 70, 72, 74, 77, 81, 88, 89, 90, 91, 95, 97, 99, 104, 107, 108, 114, 122, 124, 125, 128, 130, 134, 142, 143, 144, 146, 149, 153, 155, 158, 162, 164, 166, 173, 176, 178, 182, 184, 187
7, 8, 9, 24, 25, 28, 42, 44, 46, 61, 62, 64, 78, 79, 85, 96, 98, 103, 115, 116, 121, 133, 136, 139, 152, 154, 157, 171, 172, 175
10, 15, 18, 29, 35, 37, 49, 54, 56, 69, 73, 75, 86, 93, 94, 106, 112, 113, 126, 131, 132, 145, 150, 151, 165, 169, 170, 185, 188, 189

Algebraic automorphism mapping $\mathfrak{m}_{6.1}$ to $\mathfrak{m}_{2.1}$

$(0, 180, 80, 40, 160)(1, 174, 82, 55, 164, 16, 187, 88, 43, 159)$
$(2, 184, 76, 51, 156, 12, 179, 91, 47, 162)(3, 182, 87, 45, 153)$
$(4, 186, 84, 53, 161, 17, 173, 90, 38, 166)(5, 183, 77, 50, 158)$
$(6, 176, 83, 39, 168)(7, 172, 79, 44, 157)(8, 171, 85, 42, 154)$
$(9, 175, 78, 46, 152)(10, 185, 94, 49, 170)(11, 177, 81, 52, 167)$
$(13, 181, 89, 48, 155)(14, 178, 92, 41, 163)(15, 189, 86, 56, 169)$
$(18, 188, 93, 54, 165)(19, 66, 118, 111, 146, 34, 70, 124, 97, 141)$
$(20, 60, 120, 100, 140)(21, 71, 114, 104, 137)(22, 74, 119, 102, 148)$
$(23, 68, 122, 95, 147)(24, 64, 115, 98, 139)(25, 62, 121, 96, 136)$
$(26, 67, 117, 107, 144)(27, 57, 128, 105, 135)(28, 61, 116, 103, 133)$
$(29, 75, 131, 106, 151)(30, 65, 129, 109, 143)(31, 63, 125, 108, 134)$
$(32, 72, 123, 99, 149)(33, 58, 130, 101, 138)(35, 73, 126, 113, 150)$
$(36, 59, 127, 110, 142)(37, 69, 132, 112, 145)$

Algebraic automorphism mapping $\mathfrak{m}_{6.1}$ to $\mathfrak{m}_{7.1}$

$(0, 100, 180, 60)(1, 99, 178, 65, 16, 105, 183, 71)(2, 107, 186, 63)$
$(3, 95, 176, 70)(4, 108, 179, 58, 17, 102, 184, 67)(5, 104, 187, 57)$
$(6, 97, 182, 59)(7, 103, 172, 61)(8, 96, 175, 64)(9, 98, 171, 62)$
$(10, 112, 189, 73)(11, 110, 181, 66)(12, 101, 173, 74)(13, 111, 177, 68)$
$(14, 109, 174, 72)(15, 113, 185, 69)(18, 106, 188, 75)$
$(19, 81, 142, 155, 34, 87, 147, 168)(20, 80, 140, 160)$
$(21, 88, 135, 158, 30, 82, 149, 163)(22, 91, 138, 166, 31, 76, 144, 161)$
$(23, 89, 141, 153)(24, 85, 136, 152)(25, 78, 139, 154)$
$(26, 90, 148, 156)(27, 77, 143, 159)(28, 79, 133, 157)$
$(29, 93, 151, 165)(32, 92, 137, 164)(33, 84, 134, 162)(35, 94, 145, 169)$
$(36, 83, 146, 167)(37, 86, 150, 170)(38, 125, 51, 117)$
$(39, 124, 45, 122, 48, 118, 52, 127)(40, 120)(41, 114, 43, 128)$
$(42, 121, 46, 115)(44, 116)(47, 130, 53, 119)(49, 132, 56, 126)$
$(50, 129, 55, 123)(54, 131)$

C Data related to graph HoSi

Elements of Ω related to Proposition 3.5

0	$\emptyset$
1	0
2	1
3	2
4	3
5	4
6	5
7	6
8	{{{1, 2}, {3, 5}, {4, 6}}, {{1, 3}, {2, 6}, {4, 5}}, {{1, 4}, {2, 3}, {5, 6}}, {{1, 5}, {2, 4}, {3, 6}}, {{1, 6}, {2, 5}, {3, 4}}}
9	{{{2, 3}, {4, 6}, {5, 7}}, {{2, 4}, {3, 7}, {5, 6}}, {{2, 5}, {3, 4}, {6, 7}}, {{2, 6}, {3, 5}, {4, 7}}, {{2, 7}, {3, 6}, {4, 5}}}
10	{{{1, 2}, {3, 5}, {4, 6}}, {{1, 3}, {2, 4}, {5, 6}}, {{1, 4}, {2, 5}, {3, 6}}, {{1, 5}, {2, 6}, {3, 4}}, {{1, 6}, {2, 3}, {4, 5}}}
11	{{{1, 3}, {4, 7}, {5, 6}}, {{1, 4}, {3, 5}, {6, 7}}, {{1, 5}, {3, 7}, {4, 6}}, {{1, 6}, {3, 4}, {5, 7}}, {{1, 7}, {3, 6}, {4, 5}}}
12	{{{1, 3}, {4, 6}, {5, 7}}, {{1, 4}, {3, 7}, {5, 6}}, {{1, 5}, {3, 4}, {6, 7}}, {{1, 6}, {3, 5}, {4, 7}}, {{1, 7}, {3, 6}, {4, 5}}}
13	{{{2, 3}, {4, 6}, {5, 7}}, {{2, 4}, {3, 5}, {6, 7}}, {{2, 5}, {3, 6}, {4, 7}}, {{2, 6}, {3, 7}, {4, 5}}, {{2, 7}, {3, 4}, {5, 6}}}
14	{{{1, 2}, {4, 7}, {5, 6}}, {{1, 4}, {2, 6}, {5, 7}}, {{1, 5}, {2, 4}, {6, 7}}, {{1, 6}, {2, 7}, {4, 5}}, {{1, 7}, {2, 5}, {4, 6}}}
15	{{{2, 3}, {4, 7}, {5, 6}}, {{2, 4}, {3, 5}, {6, 7}}, {{2, 5}, {3, 7}, {4, 6}}, {{2, 6}, {3, 4}, {5, 7}}, {{2, 7}, {3, 6}, {4, 5}}}
16	{{{1, 2}, {4, 7}, {5, 6}}, {{1, 4}, {2, 5}, {6, 7}}, {{1, 5}, {2, 7}, {4, 6}}, {{1, 6}, {2, 4}, {5, 7}}, {{1, 7}, {2, 6}, {4, 5}}}
17	{{{1, 3}, {4, 5}, {6, 7}}, {{1, 4}, {3, 7}, {5, 6}}, {{1, 5}, {3, 6}, {4, 7}}, {{1, 6}, {3, 4}, {5, 7}}, {{1, 7}, {3, 5}, {4, 6}}}
18	{{{1, 3}, {4, 6}, {5, 7}}, {{1, 4}, {3, 5}, {6, 7}}, {{1, 5}, {3, 6}, {4, 7}}, {{1, 6}, {3, 7}, {4, 5}}, {{1, 7}, {3, 4}, {5, 6}}}
19	{{{1, 2}, {3, 6}, {5, 7}}, {{1, 3}, {2, 7}, {5, 6}}, {{1, 5}, {2, 3}, {6, 7}}, {{1, 6}, {2, 5}, {3, 7}}, {{1, 7}, {2, 6}, {3, 5}}}
20	{{{1, 3}, {4, 7}, {5, 6}}, {{1, 4}, {3, 6}, {5, 7}}, {{1, 5}, {3, 4}, {6, 7}}, {{1, 6}, {3, 7}, {4, 5}}, {{1, 7}, {3, 5}, {4, 6}}}
21	{{{1, 2}, {3, 7}, {5, 6}}, {{1, 3}, {2, 6}, {5, 7}}, {{1, 5}, {2, 3}, {6, 7}}, {{1, 6}, {2, 7}, {3, 5}}, {{1, 7}, {2, 5}, {3, 6}}}
22	{{{1, 2}, {4, 6}, {5, 7}}, {{1, 4}, {2, 5}, {6, 7}}, {{1, 5}, {2, 6}, {4, 7}}, {{1, 6}, {2, 7}, {4, 5}}, {{1, 7}, {2, 4}, {5, 6}}}
23	{{{2, 3}, {4, 5}, {6, 7}}, {{2, 4}, {3, 7}, {5, 6}}, {{2, 5}, {3, 6}, {4, 7}}, {{2, 6}, {3, 4}, {5, 7}}, {{2, 7}, {3, 5}, {4, 6}}}
24	{{{1, 2}, {4, 5}, {6, 7}}, {{1, 4}, {2, 7}, {5, 6}}, {{1, 5}, {2, 6}, {4, 7}}, {{1, 6}, {2, 4}, {5, 7}}, {{1, 7}, {2, 5}, {4, 6}}}
25	{{{1, 2}, {3, 7}, {4, 6}}, {{1, 3}, {2, 4}, {6, 7}}, {{1, 4}, {2, 7}, {3, 6}}, {{1, 6}, {2, 3}, {4, 7}}, {{1, 7}, {2, 6}, {3, 4}}}
26	{{{1, 2}, {3, 6}, {5, 7}}, {{1, 3}, {2, 5}, {6, 7}}, {{1, 5}, {2, 6}, {3, 7}}, {{1, 6}, {2, 7}, {3, 5}}, {{1, 7}, {2, 3}, {5, 6}}}
27	{{{1, 2}, {4, 6}, {5, 7}}, {{1, 4}, {2, 7}, {5, 6}}, {{1, 5}, {2, 4}, {6, 7}}, {{1, 6}, {2, 5}, {4, 7}}, {{1, 7}, {2, 6}, {4, 5}}}
28	{{{2, 3}, {4, 7}, {5, 6}}, {{2, 4}, {3, 6}, {5, 7}}, {{2, 5}, {3, 4}, {6, 7}}, {{2, 6}, {3, 7}, {4, 5}}, {{2, 7}, {3, 5}, {4, 6}}}
29	{{{1, 2}, {3, 6}, {4, 7}}, {{1, 3}, {2, 7}, {4, 6}}, {{1, 4}, {2, 3}, {6, 7}}, {{1, 6}, {2, 4}, {3, 7}}, {{1, 7}, {2, 6}, {3, 4}}}
30	{{{1, 2}, {3, 7}, {5, 6}}, {{1, 3}, {2, 5}, {6, 7}}, {{1, 5}, {2, 7}, {3, 6}}, {{1, 6}, {2, 3}, {5, 7}}, {{1, 7}, {2, 6}, {3, 5}}}
31	{{{1, 2}, {3, 5}, {6, 7}}, {{1, 3}, {2, 7}, {5, 6}}, {{1, 5}, {2, 6}, {3, 7}}, {{1, 6}, {2, 3}, {5, 7}}, {{1, 7}, {2, 5}, {3, 6}}}
32	{{{1, 2}, {4, 5}, {6, 7}}, {{1, 4}, {2, 6}, {5, 7}}, {{1, 5}, {2, 7}, {4, 6}}, {{1, 6}, {2, 5}, {4, 7}}, {{1, 7}, {2, 4}, {5, 6}}}
33	{{{1, 2}, {3, 7}, {4, 5}}, {{1, 3}, {2, 5}, {4, 7}}, {{1, 4}, {2, 3}, {5, 7}}, {{1, 5}, {2, 7}, {3, 4}}, {{1, 7}, {2, 4}, {3, 5}}}
34	{{{1, 2}, {3, 7}, {4, 6}}, {{1, 3}, {2, 6}, {4, 7}}, {{1, 4}, {2, 3}, {6, 7}}, {{1, 6}, {2, 7}, {3, 4}}, {{1, 7}, {2, 4}, {3, 6}}}
35	{{{1, 2}, {3, 4}, {6, 7}}, {{1, 3}, {2, 7}, {4, 6}}, {{1, 4}, {2, 6}, {3, 7}}, {{1, 6}, {2, 3}, {4, 7}}, {{1, 7}, {2, 4}, {3, 6}}}
36	{{{1, 2}, {3, 7}, {4, 5}}, {{1, 3}, {2, 4}, {5, 7}}, {{1, 4}, {2, 7}, {3, 5}}, {{1, 5}, {2, 3}, {4, 7}}, {{1, 7}, {2, 5}, {3, 4}}}
37	{{{1, 2}, {3, 6}, {4, 7}}, {{1, 3}, {2, 4}, {6, 7}}, {{1, 4}, {2, 6}, {3, 7}}, {{1, 6}, {2, 7}, {3, 4}}, {{1, 7}, {2, 3}, {4, 6}}}
38	{{{1, 2}, {3, 5}, {6, 7}}, {{1, 3}, {2, 6}, {5, 7}}, {{1, 5}, {2, 7}, {3, 6}}, {{1, 6}, {2, 5}, {3, 7}}, {{1, 7}, {2, 3}, {5, 6}}}
39	{{{1, 2}, {3, 5}, {4, 7}}, {{1, 3}, {2, 7}, {4, 5}}, {{1, 4}, {2, 3}, {5, 7}}, {{1, 5}, {2, 4}, {3, 7}}, {{1, 7}, {2, 5}, {3, 4}}}
40	{{{1, 2}, {3, 5}, {4, 7}}, {{1, 3}, {2, 4}, {5, 7}}, {{1, 4}, {2, 5}, {3, 7}}, {{1, 5}, {2, 7}, {3, 4}}, {{1, 7}, {2, 3}, {4, 5}}}
41	{{{1, 2}, {3, 4}, {6, 7}}, {{1, 3}, {2, 6}, {4, 7}}, {{1, 4}, {2, 7}, {3, 6}}, {{1, 6}, {2, 4}, {3, 7}}, {{1, 7}, {2, 3}, {4, 6}}}
42	{{{1, 2}, {3, 6}, {4, 5}}, {{1, 3}, {2, 5}, {4, 6}}, {{1, 4}, {2, 3}, {5, 6}}, {{1, 5}, {2, 6}, {3, 4}}, {{1, 6}, {2, 4}, {3, 5}}}
43	{{{1, 2}, {3, 4}, {5, 7}}, {{1, 3}, {2, 7}, {4, 5}}, {{1, 4}, {2, 5}, {3, 7}}, {{1, 5}, {2, 3}, {4, 7}}, {{1, 7}, {2, 4}, {3, 5}}}
44	{{{1, 2}, {3, 6}, {4, 5}}, {{1, 3}, {2, 4}, {5, 6}}, {{1, 4}, {2, 6}, {3, 5}}, {{1, 5}, {2, 3}, {4, 6}}, {{1, 6}, {2, 5}, {3, 4}}}
45	{{{1, 2}, {3, 4}, {5, 6}}, {{1, 3}, {2, 6}, {4, 5}}, {{1, 4}, {2, 5}, {3, 6}}, {{1, 5}, {2, 3}, {4, 6}}, {{1, 6}, {2, 4}, {3, 5}}}
46	{{{1, 2}, {3, 4}, {5, 7}}, {{1, 3}, {2, 5}, {4, 7}}, {{1, 4}, {2, 7}, {3, 5}}, {{1, 5}, {2, 4}, {3, 7}}, {{1, 7}, {2, 3}, {4, 5}}}
47	{{{1, 2}, {3, 4}, {5, 6}}, {{1, 3}, {2, 5}, {4, 6}}, {{1, 4}, {2, 6}, {3, 5}}, {{1, 5}, {2, 4}, {3, 6}}, {{1, 6}, {2, 3}, {4, 5}}}
48	{{{2, 3}, {4, 5}, {6, 7}}, {{2, 4}, {3, 6}, {5, 7}}, {{2, 5}, {3, 7}, {4, 6}}, {{2, 6}, {3, 5}, {4, 7}}, {{2, 7}, {3, 4}, {5, 6}}}
49	{{{1, 3}, {4, 5}, {6, 7}}, {{1, 4}, {3, 6}, {5, 7}}, {{1, 5}, {3, 7}, {4, 6}}, {{1, 6}, {3, 5}, {4, 7}}, {{1, 7}, {3, 4}, {5, 6}}}

List of elements of Ω (Section 11)

0	0	1	1
2	2	3	3
4	4	5	$\{\{0,5\},\{1,6\},\{2,7\},\{3,8\},\{4,9\}\}$
6	$\{\{0,9\},\{1,5\},\{2,6\},\{3,7\},\{4,8\}\}$	7	$\{\{0,8\},\{1,9\},\{2,5\},\{3,6\},\{4,7\}\}$
8	$\{\{0,7\},\{1,8\},\{2,9\},\{3,5\},\{4,6\}\}$	9	$\{\{0,6\},\{1,7\},\{2,8\},\{3,9\},\{4,5\}\}$
10	$\{\{0,5\},\{1,9\},\{2,8\},\{3,7\},\{4,6\}\}$	11	$\{\{0,9\},\{1,8\},\{2,7\},\{3,6\},\{4,5\}\}$
12	$\{\{0,8\},\{1,7\},\{2,6\},\{3,5\},\{4,9\}\}$	13	$\{\{0,7\},\{1,6\},\{2,5\},\{3,9\},\{4,8\}\}$
14	$\{\{0,6\},\{1,5\},\{2,9\},\{3,8\},\{4,7\}\}$	15	$\{\{0,5\},\{1,7\},\{2,9\},\{3,6\},\{4,8\}\}$
16	$\{\{0,8\},\{1,5\},\{2,7\},\{3,9\},\{4,6\}\}$	17	$\{\{0,6\},\{1,8\},\{2,5\},\{3,7\},\{4,9\}\}$
18	$\{\{0,9\},\{1,6\},\{2,8\},\{3,5\},\{4,7\}\}$	19	$\{\{0,7\},\{1,9\},\{2,6\},\{3,8\},\{4,5\}\}$
20	$\{\{0,5\},\{1,8\},\{2,6\},\{3,9\},\{4,7\}\}$	21	$\{\{0,8\},\{1,6\},\{2,9\},\{3,7\},\{4,5\}\}$
22	$\{\{0,6\},\{1,9\},\{2,7\},\{3,5\},\{4,8\}\}$	23	$\{\{0,9\},\{1,7\},\{2,5\},\{3,8\},\{4,6\}\}$
24	$\{\{0,7\},\{1,5\},\{2,8\},\{3,6\},\{4,9\}\}$	25	$(0,5)$
26	$(1,5)$	27	$(2,5)$
28	$(3,5)$	29	$(4,5)$
30	$(0,6)$	31	$(1,6)$
32	$(2,6)$	33	$(3,6)$
34	$(4,6)$	35	$(0,7)$
36	$(1,7)$	37	$(2,7)$
38	$(3,7)$	39	$(4,7)$
40	$(0,8)$	41	$(1,8)$
42	$(2,8)$	43	$(3,8)$
44	$(4,8)$	45	$(0,9)$
46	$(1,9)$	47	$(2,9)$
48	$(3,9)$	49	$(4,9)$

List of elements of Ω'

0	0	1	1	2	4	3	3
4	2	5	7	6	19	7	5
8	11	9	6	10	1	6 11	8
12	23	13	21	14	27	15	18
16	10	17	9	18	13	19	17
20	22	21	20	22	26	23	12
24	37	25	24	26	31	27	14
28	15	29	33	30	35	31	25
32	38	33	28	34	34	35	32
36	30	37	36	38	29	39	39
40	{0,1,2,3}	41	{4,5,6,7}	42	{16,17,18,19}	43	{8,9,10,11}
44	{20,21,22,23}	45	{24,25,26,27}	46	{12,13,14,15}	47	{36,37,38,39}
48	{28,29,30,31}	49	{32,33,34,35}				