A family of Higmanian association schemes on 40 points: A computer algebra approach

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Dedicated to Eiichi Bannai on the occasion of his 60th birthday

1 Introduction

In this paper we describe a family of imprimitive rank 5 association schemes on 40 points which are called Higmanian (they belong to class II association schemes with 4 classes according to the classification introduced by D. G. Higman in [9]). The first example of such schemes was given by Y. Chang and T. Huang in [2] based on a previous construction by A. and M. Deza [3].

A scheme $\mathfrak{M}$ considered in [2] is presented in evident form via basis matrices which are described in a block form, intersection numbers and eigenmatrices are computed, but there is no information about the automorphism group $Aut(\mathfrak{M})$. It is also shown that one of the basis graphs of $\mathfrak{M}$ is the point graph of a generalized quadrangle $Q(4,3)$. In the course of computer algebra experimentation using total graph coherent configuration with two fibers, we constructed such configuration starting from the triangular graph $T(5)$ and described all mergings of it which provide association schemes. One of the resulting fusion schemes was isomorphic to the scheme $\mathfrak{M}$ (this was first established with the aid of a computer, though later on we obtained a computer-free proof). We also found the group $G = Aut(\mathfrak{M})$ to be a certain transitive permutation group of degree 40 and order 1920. Playing with this group we have managed to establish a few nice properties of $\mathfrak{M}$ and related structures. In particular, we discovered a new partial linear space on 40 points and 40 lines of size 4 which in a sense is a "geometric generator" of all scheme $\mathfrak{M}$.

Understanding of these properties of $\mathfrak{M}$ allowed us to elaborate a nice, simple though very efficient, algorithmic approach to the constructive enumeration of all association schemes sharing with $\mathfrak{M}$ the same tensor of structure constants (algebraically isomorphic in our terminology). This approach is based on a computer inspection of the catalogue of all 28 strongly regular graphs with the parameters $(40, 12, 2, 4)$. This catalogue belongs to ES and is available from his home page.

Finally we proved that there exists precisely 15 association schemes, algebraically isomorphic to $\mathfrak{M}$. Only one of them, namely $\mathfrak{M}$ is Schurian while all the remaining schemes have intransitive automorphism group. Four of the discovered schemes, including $\mathfrak{M}$, are geometric in the sense which was explained above.

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Our results are based on use of a few computer algebra packages, namely COCO [6], GAP [17], GRAPE [18], and nauty [15], although we finally managed to provide computer-free proofs for the most important (and beautiful in our eyes) results.

2 Preliminaries

Association schemes

A *coherent algebra* is a matrix algebra which is closed with respect to SH (elementwise) multiplication and transposition, and contains the matrices I and J. The rank of a coherent algebra is its dimension. A *coherent configuration* is a relational reformulation of coherent algebra. That is, a coherent configuration can be defined as a set of relations such that their adjacency matrices form a basis of a coherent algebra. We refer to [5], [8], [13] for more information about coherent configurations (algebras) and association schemes. An *association scheme* is a coherent configuration for which all basis relations are regular.

Generalized quadrangles of order 3

A generalized quadrangle is a partial geometry $PG(s,t,1)$, i.e., an incidence structure for which every block has $s+1$ points, every point lies on $t+1$ blocks, two blocks intersect in at most one point, and for every block b and every point P not in b, there is exactly one block c such that P is in c, and b intersects c.

A generalized quadrangle of order 3 is $PG(3,3,1)$, denoted by $GQ(3)$. A $GQ(3)$ has 40 points and 40 blocks.

In [16] it is shown that up to isomorphism there are two generalized quadrangles of order 3, $W(3)$ and $Q(4,3)$.

3 Imprimitive association schemes of low rank

The smallest rank for which non-trivial imprimitive association schemes exist is equal to 4. The paper [20] provides a nice survey of various classes of such schemes.

Rank 5 imprimitive schemes have been investigated with less attention. A general program was outlined by Higman in [9].

Let E be an equivalence relation in an association scheme $\mathfrak{M}$ (closed subset in Zieschang's terms, parabolic in terms of Higman). Then $rank(E)$ is the rank of the association scheme induced by $\mathfrak{M}$ on one (any) of the equivalence classes of E, while $corank(E)$ is the rank of the quotient association scheme $\mathfrak{M}/E$.

It is easy to see that $rank(E) + corank(E) \leq rank(\mathfrak{M}) + 1$, with equality if and only if $\mathfrak{M}$ is the wreath product of E and $\mathfrak{M}/E$.

This makes it reasonable to consider the following classes (due to Higman) of rank 5 imprimitive symmetric association schemes with a parabolic E which are not decomposable into a wreath product:

class of $\mathfrak{M}$	I	II	III
rank of E	3	2	2
corank of E	2	3	2

In [9] examples are given, which show that classes I and II may have non-trivial intersection, while class III is distinct from classes I and II. Class I was investigated extensively by Higman.

Let E have n equivalence classes of size v, and suppose that one of two associate classes of $\mathfrak{M}/E$ is a strongly regular graph with the parameters (n, k, l, λ, μ). Then the whole scheme $\mathfrak{M}$ has nv points and valencies $k_0 = 1$, $k_1 = v - 1$, $k_2 = kS$, $k_3 = (v - S)k$, $k_4 = lv$ for a suitable parameter S. Corresponding intersection matrices and character tables are provided in [9], as well as for a particular case, which corresponds to the intersection of classes I and II. Higman refers to examples of distance regular imprimitive graphs of diameter 4 (see, e.g. [1]), briefly discussing when these graphs also belong to class I.

We are interested in rank 5 schemes which belong properly to class II. Such an example was provided in [2]. It has 40 points and corresponds to the following Higman parameters: $n = 10$, $k = 6$, $l = 3$, $\lambda = 3$, $\mu = 4$ (that is the complement of the Petersen graph), $v = 4$, $S = 2$, thus resulting in four classes with the valencies 3, 12, 12, 12.

In this paper we provide detailed investigation of this example and classify all association schemes which are algebraically isomorphic to it.

4 Starting point: Total configuration of $T(5)$.

Let $\Sigma = (V, E)$ be a graph. The total graph $T(\Sigma)$ is the graph with the vertex set $V \cup E$, two such vertices in $T(\Sigma)$ are adjacent if and only if they are adjacent or incident in Σ (here edges of Σ are incident if they have a joint vertex).

A coherent closure of $T(\Sigma)$ will be called a *total coherent configuration* of Σ.

We are interested in the total coherent configuration $\mathfrak{T}(m)$ of the triangular graph $T(m)$ (recall that $T(m)$ is the line graph $L(K_m)$ of the complete graph K_m). Clearly, $\mathfrak{T}(m)$ is a coherent configuration with 2 fibers on $\frac{m(m-1)^2}{2}$ points.

The first non-trivial case corresponds to $m = 4$. Here (according to COCO) we get a coherent configuration of rank 18 which has a few Schurian mergings of rank 3 and 4. All these mergings are quite predictable.

The first surprises appear in the case $m = 5$. Here we get a coherent configuration of rank 24 with 2 fibers of size 10 and 30.

COCO returns 9 mergings, all Schurian, among them 4 association schemes of rank 5 and one primitive strongly regular graph with the parameters $(40, 12, 2, 4)$ and rank 3 automorphism group of order 51840.

Two of the above association schemes with 4 classes have valencies 1, 3, 12, 12, 12. With the aid of GAP we prove that they are both isomorphic to the Higmanian association scheme from [2].

For these schemes one of the classes of valency 12 provides the above rank 3 graph, which should be the point graph of a generalized quadrangle $GQ(3)$. According to [2], this graph is the polar graph $O_5(3)$, in other words the point graph of $Q(4,3)$.

As was mentioned, the story of our association scheme with 4 classes goes back to the paper [3], in which the ridge graph Γ_5 was defined. (Note that one more description of Γ_5 with correction of some misprints in [3] appears in [4].) This ridge graph has valency 15 and is easily described via the union of one relation of valency 12 with the relation of valency 3 in the Higmanian association scheme.

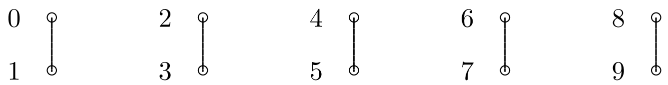
5 The group of order 1920 and its actions

One of the main paradigms in our vision of computer algebra experimentation in algebraic combinatorics may be formulated as follows:

In order to understand properly a combinatorial object O in consideration, describe its automorphism group $G = Aut(O)$ and reveal all actions of G which should be naturally attributed to O. Sometimes it may be very helpful to start from a certain auxiliary structure Δ and to define the action of G on this structure.

In this text we are following the formulated paradigm.

Let us consider graph $\Delta = 5 \circ K_2$ as follows:



The automorphism group $Aut(\Delta)$ is isomorphic to the wreath product $S_5 \wr S_2$ of order $5! \cdot 25 = 3840$. We refer to [5] and [13], where the operation of the wreath product of groups is treated in notation and style consistent with our current presentation.

The group $Aut(\Delta)$ clearly contains odd permutations. This is why the group $G = (S_5 \wr S_2)^{pos}$ which consists of all even permutations from $Aut(\Delta)$ has index 2 in $Aut(\Delta)$. It is, in a sense, our master group in this paper.

For the sake of convenience (especially using COCO) in what follows we will consider G defined with the aid of the following generators:

$$G =< (0,6,4,1,7,5)(8,9), (0,7,1,6)(2,5,8,3,4,9) > .$$

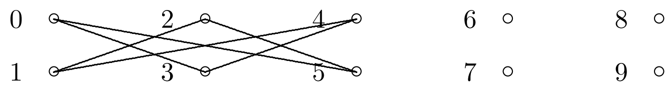
We now wish to describe in terms of Δ the action of G on the points of a new model of $Q(4,3)$ (to be presented in Section 6), as well as on the points of the Higmanian association scheme $\mathfrak{M}$. The fact that G is indeed isomorphic to the group $Aut(\mathfrak{M})$ can be confirmed by GAP. GAP returns also the description of a point stabilizer in this transitive action, namely a group H_1 of order 48 isomorphic to $D_6 \times E_4$, that is the direct product of dihedral group D_6 of order 12 and the elementary abelian group of order 4.

We get the following set of generators for H_1:

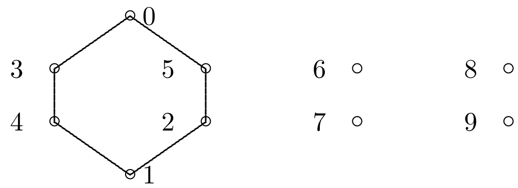
$$H_1 = \left\langle \begin{array}{c} (6,7)(8,9), (0,1)(2,3)(4,5)(8,9), \\ (6,8)(7,9), (0,2,4)(1,3,5), (2,4)(3,5) \end{array} \right\rangle .$$

In fact, COCO requires to interpret H_1 as the intersection of the group G with the automorphism group of a suitable structure (or string), say $S^{(1)}$, which is defined in terms of auxiliary structure Δ. Then the required set Ω of points of the model for $\mathfrak{M}$ will be obtained via induced action of G on the images of $S^{(1)}$.

Let us take as the role of $S^{(1)}$ the following hexagon:



or, in a more beautiful form:



Clearly our group H_1 stabilizes $S^{(1)}$. On the other hand stabilizer of $S^{(1)}$ in $Aut(\Delta)$ has order $12 \cdot 8 = 96$ and contains odd permutations, for example $(6,7)$. Therefore $Aut(S^{(1)}) \cap G = H_1$. Define $\Omega = {S^{(1)}}^G$ and obtain that $|\Omega| = [G : H_1] = \frac{1920}{48} = 40$. (List of elements of Ω is attached in archive file which may be provided by authors.)

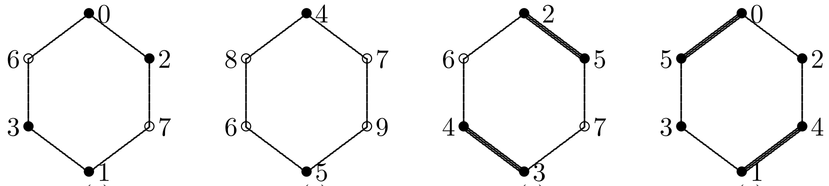
It is easy to see also that the complementary graph $\overline{\Delta} = \overline{5 \circ K_5}$ has $\binom{5}{2}4 = 40$ inscribed subgraphs which are isomorphic to the hexagon $S^{(1)}$. Therefore the required set Ω may also be described purely combinatorially.

We use COCO and get a Schurian association scheme $(\Omega, 2-orb(G,\Omega))$. It has rank 5 with classes of valency 12, 12, 12, 3. The group G is its full automorphism group. (Labeling of classes is produced by COCO.) We also get a description of structure constants and of all fusion schemes. GAP confirms that $(\Omega, 2-orb(G,\Omega))$ is isomorphic to the original Higmanian association scheme. Thus from now we simply write

$$\mathfrak{M} = (\Omega, 2-orb(G,\Omega)).$$

Let us describe this scheme in a more friendly form providing computer-free proofs of some of its properties, when this seems to be reasonable and productive.

Therefore from now we attribute the label $S^{(1)}_0$ to the above copy of the hexagon (for brevity, four isolated points are omitted), and following COCO, depict a few other requested copies of structure $S^{(1)}$:



$S_1^{(1)}$ $S_2^{(1)}$ $S_3^{(1)}$ $S_7^{(1)}$

With the aid of these representatives of Ω we define basis relations on $\mathfrak{M}$ as follows: R_0, R_1, R_2, R_3, R_4, where $R_i = (S_0^{(1)}, S_{j_i}^{(1)})^G$, $j_i = 0, 1, 2, 3, 7$ respectively.

We now give more formal description of the relations in which we regard $S_0^{(0)}$ as the reference copy. Then black vertices and bold edges on other hexagons reveal their intersection with the reference copy. Using the methodology of subsequent splitting of relations (see [5]) we may easily distinguish basis relations of $\mathfrak{M}$ via two invariants of pairs of hexagons:

R_i	R_0	R_1	R_2	R_3	R_4
Nr. of joint points	6	4	2	4	6
Nr. of joint edges	6	0	0	2	2

It is immediately clear from the description that $E = R_0 \cup R_4$ is an equivalence relation with 10 classes of size 4 parametrized by 2-element subsets of the edges of Δ. Thus we may define quotient graphs for all the remaining basis graphs Γ_1, Γ_2, Γ_3. Simple combinatorial arguments show that Γ_2/E is isomorphic to the Petersen graph, while Γ_1/E and Γ_3/E are isomorphic to its complement. Γ_1 is the point graph of $Q(4,3)$ (a computer-free proof of this result will be discussed in next section).

From this information it follows that Γ_2 is the wreath product graph.

COCO also returns the following complete list of mergings:

$\mathfrak{M}_1 = (\Omega, \{R_0, R_1 \cup R_3, R_2, R_4\})$,

$\mathfrak{M}_2 = (\Omega, \{R_0, R_1 \cup R_2 \cup R_3, R_4\})$,

$\mathfrak{M}_3 = (\Omega, \{R_0, R_1, R_2 \cup R_3 \cup R_4\})$.

Analyzing this list we detect that the only subalgebra of the adjacency algebra $\mathfrak{A}$ of $\mathfrak{M}$, which contains the basis matrix A_3 is $\mathfrak{A}$ itself. In other words, $\mathfrak{A}$ is generated by A_3. We denote this fact as $\mathfrak{A} = << A_3 >>$, where $<< A_3 >>$ stands for the coherent closure of A_3 (see references in section 2).

Summing up all detected information we obtain:

Proposition 5.1. *a)* $\mathfrak{M} = (\Omega, 2 - orb(G, \Omega))$ *is an association scheme of rank 5 with valencies 1, 12, 12, 12, 3;*

b) $Aut(\mathfrak{M}) = G$*;*

c) $E = R_0 \cup R_4$ *is the unique closed subset in* $\mathfrak{M}$*;*

d) The quotient scheme $\mathfrak{M}/E$ is isomorphic to the rank 3 association scheme $\mathfrak{J}(5,2)$;

e) $\mathfrak{M}_1 \cong \mathfrak{J}(5,2) \wr W(K_4)$ ($W(K_4)$ is coherent closure of K_4);

f) $\mathfrak{M}_3$ is 2-class association scheme corresponding to $Q(4,3)$;

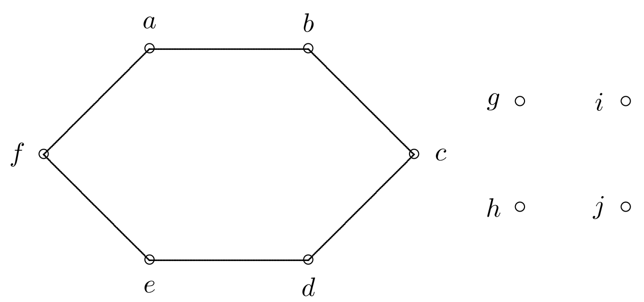
g) $\mathfrak{M}$ belongs to class II of association schemes of rank 5 in notation of Higman;

h) Up to isomorphism, $\mathfrak{M}$ is the scheme which was introduced in [2].

In what follows we will call the basis graph Γ_3 of $\mathfrak{M}$ the *classical Higmanian graph* of valency 12 on 40 points.

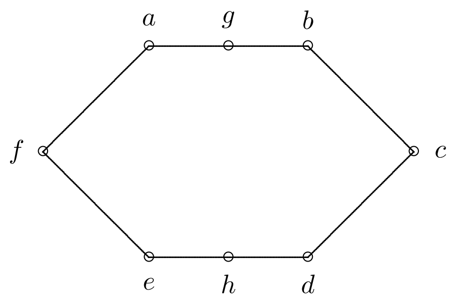
We want to give to this graph a more transparent description, because in a sense description of Γ_3 implies description of the whole scheme $\mathfrak{M}$. For this purpose we will use methodology of so-called reaction graphs, see e.g. [14], [12], [11].

Let $S \in \Omega$ be an arbitrary hexagon, described as follows:

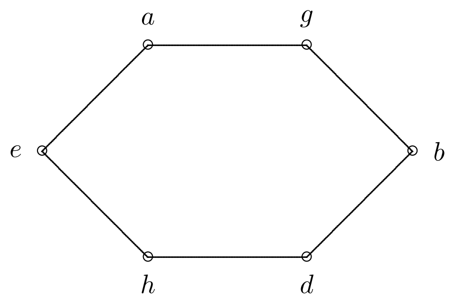


Let us consider the following transformation of S:

- select a pair of opposite edges;
- put on each edge one of the vertices g and h;
- get auxiliary graph S' with 8 vertices;



- get a homeomorphic image S'' of S' contracting vertices f and c.



We will call *reaction on* S an operation of transformation from S to S''. Clearly such reaction can be arranged by $3 \cdot 4 = 12$ different opportunities.

We now define the *reaction graph* Γ with the vertex set Ω'. Here Ω' is the set of all hexagons which may be obtained in a few reaction steps from a prescribed copy S of hexagon, say from $S_0^{(1)}$. Two hexagons from Ω' are joined by an edge if the second is obtained from the first via a reaction as above. The following lemma may be proved with the aid of (slightly routine) hand or computer considerations.

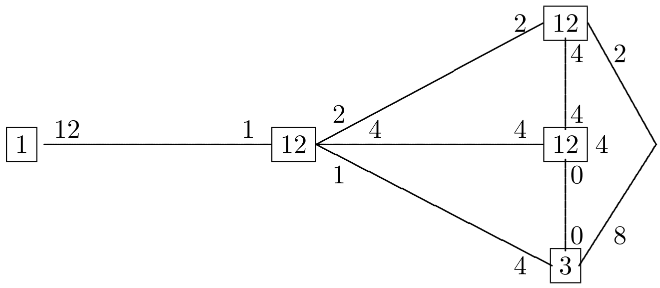
Lemma 5.2. *a) For each* $S \in \Omega$, $S'' \in \Omega$;

b) $\Omega' = \Omega$;

c) Γ *is connected graph of valency 12;*

d) Γ *has diameter 2 and girth 3;*

e) the intersection diagram of Γ *looks as*



f) Γ *is isomorphic to the Higmanian graph* Γ_3 *as above;*

g) $Aut(\Gamma) = G$;

h) Γ *is locally* $2 \circ P_6$, *the disjoint union of two prisms* P_6 *with 6 vertices.*

In what follows we will exploit various consequences of this lemma. In particular, knowledge of the intersection diagram allows us to calculate for $A = A(\Gamma)$ its square A^2 and cube A^3 and to describe the multiplication table in the adjacency algebra $\mathfrak{A}$.

Another approach to the investigation of $\mathfrak{M}$ concerns the consideration of certain incidence structures. For this purpose we need to get more information about certain subgroups of G.

Using GAP we describe all (up to conjugacy) subgroups of G. Among them we reveal 14 conjugacy classes of subgroups of order 48. One of these classes with the representative H_1 was already submitted earlier. This is stabilizer of a point in the transitive action (G, Ω).

Using GAP, we describe all orbits of (G, Ω) on the set $\left\{ \begin{matrix} \Omega \\ 4 \end{matrix} \right\}$ of the 4-element subsets of Ω. It turns out that there are 94 such orbits with lengths from 10 (it corresponds to spread, that is graph Γ_4) to 1920. Two of these orbits are of a special interest, because they have the desired length 40. In other words, the stabilizer of a corresponding 4-subset is a subgroup of order 48 in G.

Let us now describe these selected subgroups. Subgroup H_2 as an abstract group is isomorphic to the group $GL(2,3)$. It can be defined as

$$H_2 = \left\langle \begin{matrix} (2,3)(4,5)(6,7)(8,9), & (4,6,8)(5,7,9), \\ (2,8,3,9)(4,6,5,7), & (0,1)(4,5)(6,9)(7,8) \\ (2,4,3,5)(6,8,7,9) & \end{matrix} \right\rangle .$$

Let us now consider the following structure M:

$$M = \left\{ \begin{matrix} \{2,4,7\}, & \{2,6,9\}, & \{2,5,8\}, & \{4,6,8\}, \\ \{3,5,6\}, & \{3,7,8\}, & \{3,4,9\}, & \{5,7,9\} \end{matrix} \right\} .$$

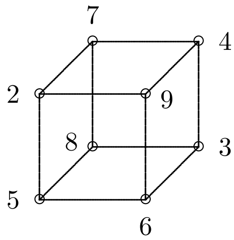
It is easy to see that M is a partial linear space with the point set $\{2,3,4,5,6,7,8,9\}$ (in fact this is a copy of a classical configuration 8_3.) Also we can check that the stabilizer of the set M in the group G coincides with the group H_2.

We also introduce the group H_3, which as an abstract group is isomorphic to $S_4 \times S_2$. Namely

$$H_3 = \left\langle \begin{matrix} (2,3)(4,5)(6,7)(8,9), & (4,6,8)(5,7,9), \\ (2,4)(3,5)(6,8)(7,9), & (6,8)(7,9), \\ (2,6)(3,7)(4,8)(5,9) & \end{matrix} \right\rangle .$$

Again we can easily check that H_3 is the automorphism group of the cube Q_3 below. Moreover this cube is an inscribed subgraph of $\overline{\Delta}$, and the stabilizer of Q_3 in G

coincides exactly with H_3.



Groups H_1, H_2 and H_3 play a significant role in the next section.

6 Two partial linear spaces on 40 points

First, we will discuss our new model for the generalized quadrangle $Q(4, 3)$.

Let us consider an incidence structure $\mathfrak{I}_1 = (P, \mathfrak{L}_1)$, where $P = \Omega$ is the set of 40 inscribed hexagons of $\overline{\Delta}$, as it is defined in the previous section. We define $\mathfrak{L}_1 = M^G$ as the orbit of a copy of a partial linear space M under the action of G.

Incidence is defined in the following manner: Each copy $M_i \in \mathfrak{L}_1$ contains 4 pairs of opposite triangles. A hexagon from P is matched to a pair of opposite triangles, if they both have the same 6 vertices, and no shared edges (so in a sense, their union, together with the edges from the original Δ, is the complete graph). For example, a pair of opposite triangles of M: $\{2, 4, 7\}, \{3, 5, 6\}$ matches the hexagon $(2, 5, 7, 3, 4, 6)$. A hexagon from P is incident to M_i if it matches a pair of opposite triangles of M_i.

Proposition 6.1. *a) The incidence structure $\mathfrak{I}_1$ has $v = 40$ points, $b = 40$ lines, each line has $k = 4$ points and each point is on $r = 4$ lines;*

b) $\mathfrak{I}_1$ is a partial linear space;

c) $\mathfrak{I}_1$ is a model of a generalized quadrangle $GQ(3)$;

d) $\mathfrak{I}_1$ is isomorphic to $Q(4, 3)$;

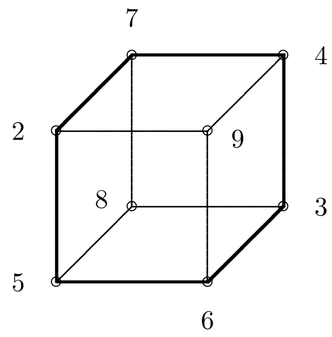
e) $Aut(\mathfrak{I}_1)$ has order 51840.

We refer to [22] for a proof of this proposition.

Now we intend to consider a model for another incidence structure formed by vertices and 4-cliques of Γ_3.

We have the same point set $P = \Omega$. The set of lines $\mathfrak{L}_2$ is the orbit Q_3^G of a selected copy of Q_3 under the action of group G.

Let us look at this copy from a different point of view. Namely, consider the following diagram of the same Q_3:



It shows a hexagon from Ω, $(2,5,6,3,4,7)$ inscribed into this copy of Q_3.

Formal definition: remove a pair of antipodal vertices $\{8,9\}$ from Q_3 and consider the subgraph induced by the remaining vertices. This is a copy of hexagon. Note that this hexagon is the *automorphic subgraph* of Q_3 (see [10]), that is the stabilizer of the hexagon in $Aut(Q_3)$ is equal to the full automorphism group D_6 (of order 12) of the hexagon.

Clearly for a fixed copy of Q_3 the incidence just introduced may be established in precisely 4 different ways, that is removing different antipodal pairs of vertices.

Proposition 6.2. *a)* $v = b = 40$*,* $k = r = 4$*;*

b) $\mathfrak{I}_2$ *is a partial linear space;*

c) Γ_3 *is the point graph of* $\mathfrak{I}_2$*;*

d) $\mathfrak{I}_2$ *is uniquely reconstructed from* Γ_3*;*

e) $Aut(\Gamma_3) = Aut(\mathfrak{I}_2) = G$*;*

f) for each line $l \in \mathfrak{L}$ *there are precisely:*

- *12 points* $P \notin l$ *through which there are 0 lines intersecting* l*;*
- *12 points* $P \notin l$ *through which there are 1 lines intersecting* l*;*
- *12 points* $P \notin l$ *through which there are 2 lines intersecting* l*;*

7 Search for all schemes algebraically isomorphic to $\mathfrak{M}$

The problem posed in the title of the section was solved with the aid of a computer.

The starting point was a catalogue of all strongly regular graphs with the parameters $(40, 12, 2, 4)$, which was produced by ES, see [19]. There are precisely

28 non-isomorphic such graphs. Note that significant portion of those graphs was already described by W. H. Haemers in [7]. Haemers was using methodology developed in Moscow by V. L. Arlazarov et al (see [21]).

The computer search was organized as follows:

- Consider each strongly regular graph $\overline{\Gamma}$ from the catalogue ($\overline{\Gamma}$ has valency 27).
- Describe all orbits of cliques of size 8 in $\overline{\Gamma}$.
- Use detected cliques as possible "hyperedges" in a wreath product $W = Petersen\ wr\ K_4$ (this is a regular graph of valency 15). Classify all (up to automorphisms from $Aut(\Gamma)$) possible embeddings of W into $\overline{\Gamma}$.
- Consider difference $\overline{\Gamma}\backslash W$, which is a regular graph of valency 12 (candidate to be an analogue of classical Higmanian graph). Find coherent closure of $\overline{\Gamma}\backslash W$. Disregard result if it has rank larger than 5.
- In case when coherent closure has rank 5 check if it is algebraically isomorphic to the Higmanian scheme $\mathfrak{M}$.

This algorithm was programmed in GAP with the aid of GRAPE.

In what follows we are using labeling of strongly regular graphs as in [19].

It turns out that precisely first 11 graphs have cliques of size 8. All these graphs admit at least one Higmanian association scheme. Altogether we get 15 schemes.

8 Survey of computer results

The main results of computation are presented in table form below. Here we show for each graph order of its automorphism group and lengths of its orbits, similar information is provided for automorphism group of each association scheme.

Note that now the classical Higmanian association scheme coincides with the scheme $\mathfrak{M}_{6.1}$.

We provide also information about the number of 4-cliques in each of 15 Higmanian graphs. An interesting correlation appears with the property to be geometric. Only those (4) Higmanian graphs are the point graphs of a suitable partial linear space, which admit (like classical Higmanian graph) precisely 40 cliques of size 4.

Γ_i	$\mathfrak{M}_i$	$\lvert Aut(\Gamma)\rvert$	$\lvert orb(Aut(\Gamma))\rvert$	$\lvert Aut(\mathfrak{M})\rvert$	$\lvert orb(Aut(\mathfrak{M}))\rvert$	Geometric	Nr. of 4-cliques
Γ_1	1.1	48	$4, 12^3$	48	same	no	32
Γ_2	2.1	384	$16, 24$	384	same	no	8
Γ_2	2.2	384	$16, 24$	192	same	yes	40
Γ_3	3.1	8	$2^8, 4^2, 8^2$	8	same	no	20
Γ_4	4.1	12	$1, 3^3, 6^3, 12$	12	same	no	24
Γ_5	5.1	64	$8, 16^2$	64	same	yes	40
Γ_5	5.2	64	$8, 16^2$	32	$4^2, 8^2, 16$	no	24
Γ_6	6.1	51840	40	1920	same	yes	40
Γ_7	7.1	192	$4, 12, 24$	192	same	no	24
Γ_7	7.2	192	$4, 12, 24$	32	$4^2, 8^2, 16$	yes	40
Γ_8	8.1	8	$2^8, 4^2, 8^2$	8	same	no	28
Γ_9	9.1	48	$2, 4, 6, 12, 16$	16	$2^4, 4^4, 16$	no	32
Γ_{10}	10.1	16	$4^2, 8^4$	16	same	no	32
Γ_{10}	10.2	16	$4^2, 8^4$	8	$4^8, 8$	no	32
Γ_{11}	11.1	144	$4, 12, 24$	48	same	no	32

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