A STORY OF AN "ARCHITECTURALLY SUGGESTIVE" POLYHEDRON: FROM MEDIEVAL TRADE TO RENAISSANCE ART AND MODERN DESIGN

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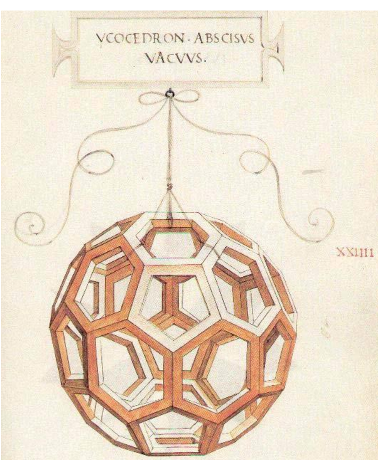
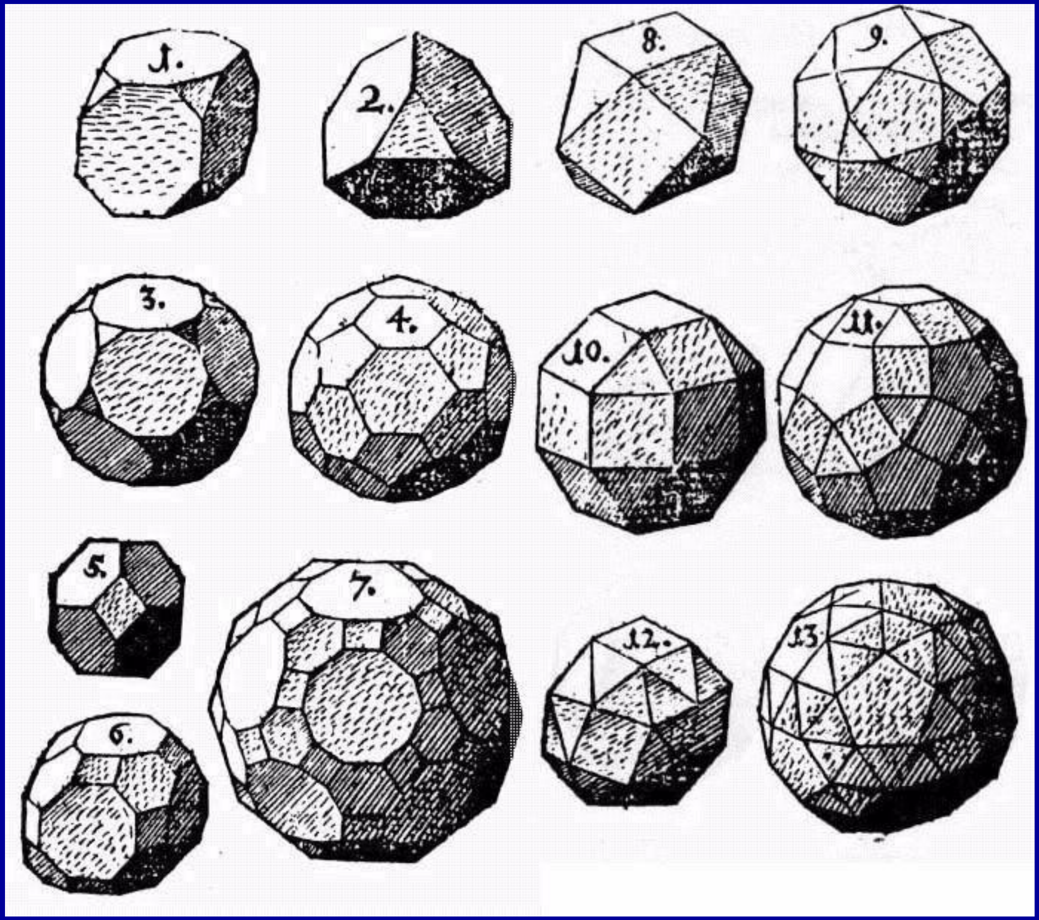


Leonardo, 1509

A. Dürer, 1497-98

Archimedes (287-212 b.c.) - first researcher of truncated icosahedron



Archimedian polyhedra or *archimedean solids*:
polyhedra with faces as regular polygons and vertices located in equivalent positions.
Archimedean solids consist of at least 2 different types of polygons (that makes them different from *regular polyhedra* or *platonic solids)*

From Johannes Kepler's "Harmonices Mundi" ("The Harmony of the World"),

1619

Platonic solids (regular polyhedra)

Classical elements:

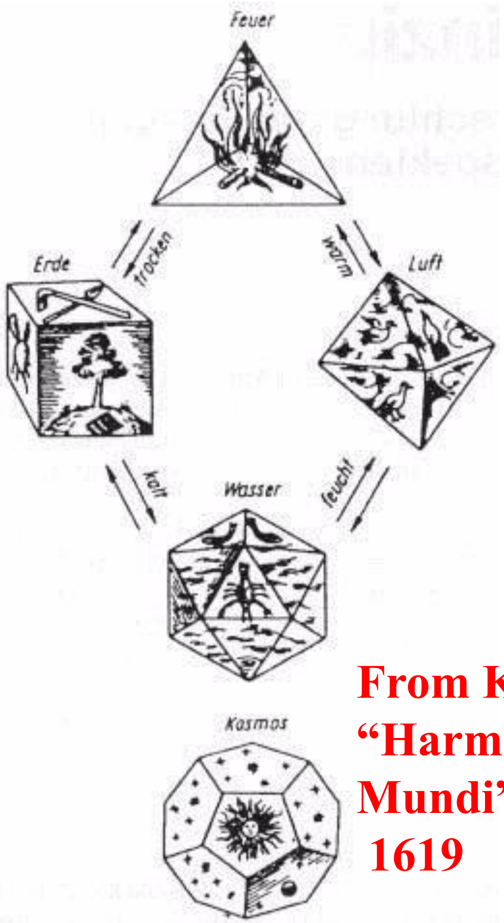
Plato (~420 – 347 b.c.), probably, Pithagoreans

Tetrahedron - fire,
Cube - earth,
Octahedron - air,
Icosahedron - water,

Empedocles (494 – 434 BC)

Hipparchus (? – 127 b.c.) Dodecahedron the universe (ether)

Aristotle (384–322 BC)



From Kepler's "Harmonice Mundi", 1619

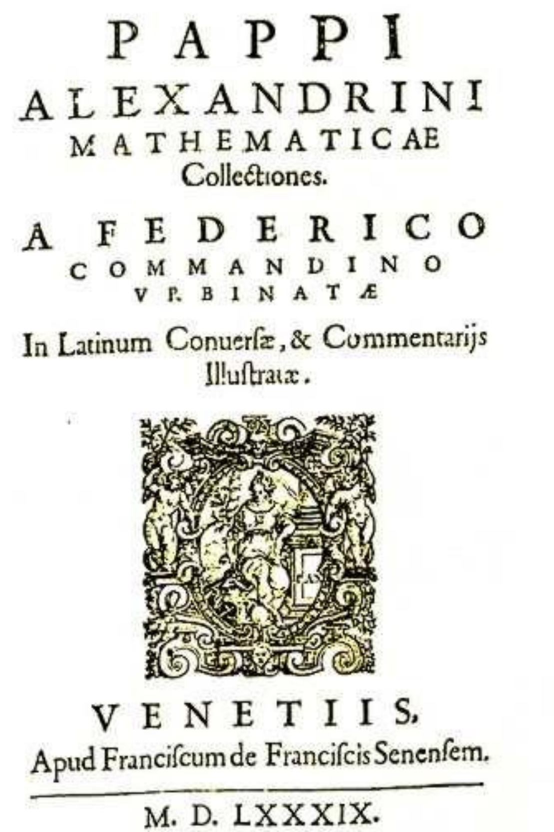
Platonic Polyhedra	Classical elements (essences)	States of matter (modern view)	Basic concepts of reality (modern paradigm)
Tetrahedron	Fire	Plasma	Space
Cube	Earth	Solid	Time
Octahedron	Air	Liquid	Energy
Icosahedron	Water	Gas	Matter
Dodecahedron	The Universe (Quintessence)		~~Ether~~ Light (Quintessence)

$E = mc^2$

Archimedes.
Ugo da Carpi (c. 1450–1480 – c. 1523–1532), after Raphael. , 1518-20, Albertina, Vienna.

Rhombicuboctahedron

The manuscript of Archimedes is generally thought to have been lost on the famous conflagration of the Ancient Library of Alexandria. There was a reference to this manuscript and description of Archimedean solids in the book (actually this is a collection of 8 books) of one of the last great Greek mathematicians of antiquity Pappus of Alexandria (290 – 350) “Synagoge” “or “Collection” (“Synagoge” is “Collection” in Greek, συναγωγή). Alexander Jones wrote: "The only ancient source for these solids [the Archimedean solids] is Pappus Book 5, chapters 34-37, together with a marginal note describing the construction of some of them”. One cannot exclude that some knowledge about Book V of “Collection” was spread in Europe in the XIII-XV centuries. It is clear however that the wide knowledge of this book had to wait until the end of XVI century when its Latin translation was published.



PAPPI
ALEXANDRINI
MATHEMATICAE
Collectiones.

A FEDERICO
COMMANDINO
VRBINATE
In Latinum Conuerſæ, & Commentarijs
Illuſtratæ.

VENETIIS,
Apud Franciſcum de Franciſcis Senenſem.

M. D. LXXXIX.

Title page of Pappus's *Mathematicae Collectiones*, translated into Latin by Federico Commandino (1589).
From wiki

Raphael (1483-1520). Fresco "The School of Athens", 1509 – 1511. Stanze di Raffaello, the Apostolic Palace, the Vatican

Plato (Leonardo)   Aristotle

Euclid (around 300 BC), the "father of geometry".

“The Elements” is a mathematical treatise consisting of 13 books.

It is a collection of definitions, postulates, propositions (theorems and constructions), and mathematical proofs of the propositions.

Book 13 constructs *the five regular Platonic solids.*

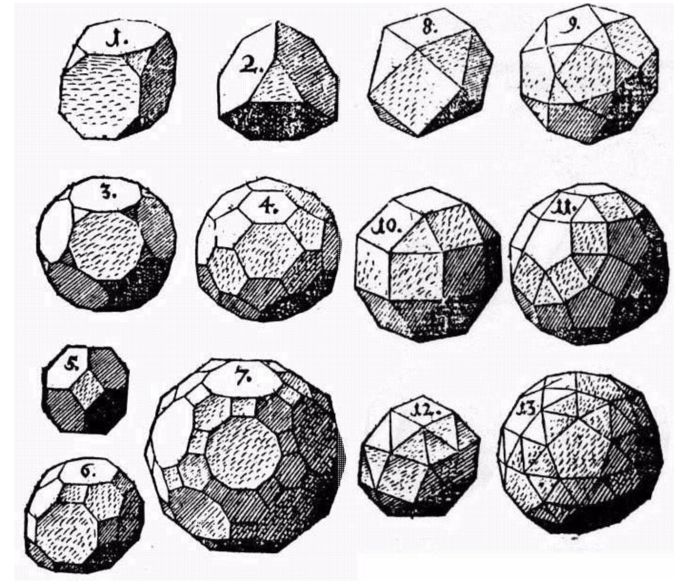
D’Arcy Thompson: *“Euclid never dreamed of writing an Elementary Geometry.*
What Euclid really did was to write a very excellent account of the regular solids, for the use of Initiates.”

Raphael.
The School of Athens
(1509–1511)

al-Ḥajjāj b. Yūsuf b. Maṭar (786-833), a mathematician and translator active in Baghdad, the author of first two translations Euclid's Elements from Greek into Arabic.
At the beginning of the 12th century CE, Adelard of Bath translated the second al-Ḥajjāj's version of Euclid's Elements (813–833) into Latin.

Isḥāq ibn Ḥunayn (830-910), influential Arab physician and translator, known for writing the first biography of physicians in the Arabic language. He is also known for his translations of Euclid's Elements and Ptolemy's Almagest. He is the son of the famous translator Hunayn Ibn Ishaq (809–873, Arab Nestorian Christian scholar, physician, and scientist, “the father of Arab translations”, translator of 116 works, including Plato's Timaeus, Aristotle's Metaphysics, and the Old Testament, into Syriac and Arabic).

1. Zeno. 2.Epicourt. 3.Frederico II, Duke of Mantua. 4. Anaximander or Empedocles of Akragant. 5. Averroes (also known as Ibn Rushd). 6. Pythagoras. 7.Alcibiades or Alexander the Great. 8. Antisthenes or Xenophon. 9. Hypatia (portrait of Raphael's mistress and model - Margherita). 10. Aeschines or Xenophon. 11. Parmenides. 12. Socrates. 13. Heraclitus of Ephesus (portrait resemblance to Michelangelo). 14. Plato (portrait of Leonardo da Vinci) with the 'Timaeus' in his left hand. 15. Aristotle, holding 'Ethics'. 16. Diogenes. 17 Plotinus. 18. Euclid (portrait resemblance to the architect Bramante). 19. Most likely, Hipparchus, or Strabo or Zarathustra. 20. Claudius Ptolemy. 21 Protogen (Perugino / Sodoma). R. Apelles (Raphael)



Medieval and the Renaissance history of Archimedean polyhedra (Archimedean solids)

Characteristics of the Archimedean polyhedra

Archimedean Solids	Number of			Described by
	Vertices	Faces[a]	Edges	
Truncated tetrahedron	12	$4_3 + 4_6 = 8$	18	Piero della Francesca
Cuboctahedron	12	$8_3 + 6_4 = 14$	24	Piero della Francesca
Truncated cube	24	$8_3 + 6_8 = 14$	36	Piero della Francesca
Truncated octahedron	24	$6_4 + 8_6 = 12$	36	Piero della Francesca
Rhombicuboctahedron	24	$8_3 + 6_4 + 12_4 = 26$	48	Luca Pacioli
Snub cube	24	$24_3 + 8_3 + 6_4 = 38$	60	Albrecht Dürer
Icosidodecahedron	30	$20_3 + 12_5 = 32$	60	Luca Pacioli
Truncated cuboctahedron	48	$12_4 + 8_6 + 6_8 = 26$	72	Albrecht Dürer
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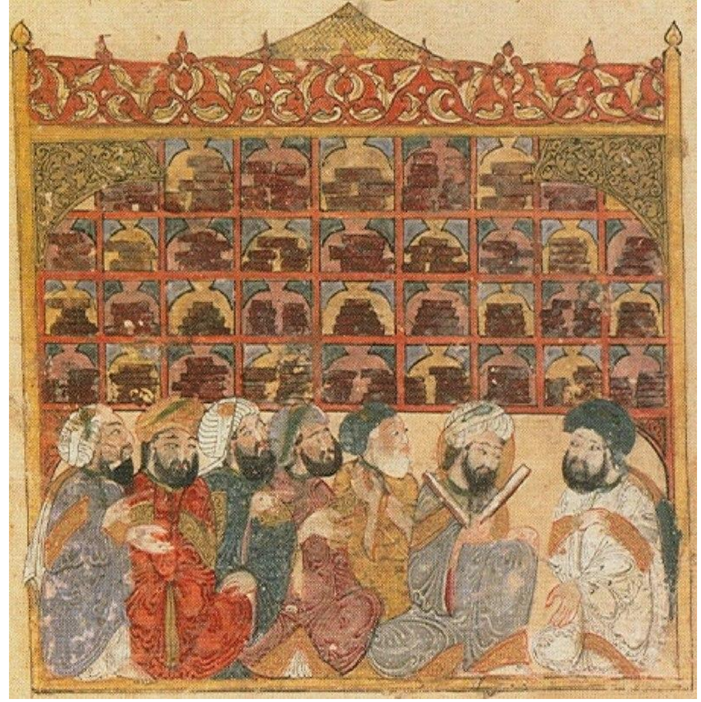
Snub dodecahedron[b]	60	$60_3 + 20_3 + 12_5 = 92$	150	Johannes Kepler
Truncated icosidodecahedron	120	$30_4 + 20_6 + 12_{10} = 62$	180	Daniele Barbaro (1514 - 1570)

Knowledge of ancient Greek mathematics, such as Euclid and Ptolemy, was lost to the medieval West, but Islamic scholars preserved their writings in Arabic translations. In the 8th-9th century, caliphs established the House of Wisdom in Baghdad as a place for scholars to acquire and translate foreign texts in mathematics and philosophy. Ptolemy's thirteen-volume work is known today by the name they gave it, Almagest (Arabic for "the greatest").

From: L. Gamwell. Why the history of maths is also the history of art. The Guardian (2.12.2015, https://www.theguardian.com/science/alexs-adventures-in-numberland/2015/dec/02/why-the-history-of-maths-is-also-the-history-of-art). Review on the book "Mathematics and Art" by L. Gamwell

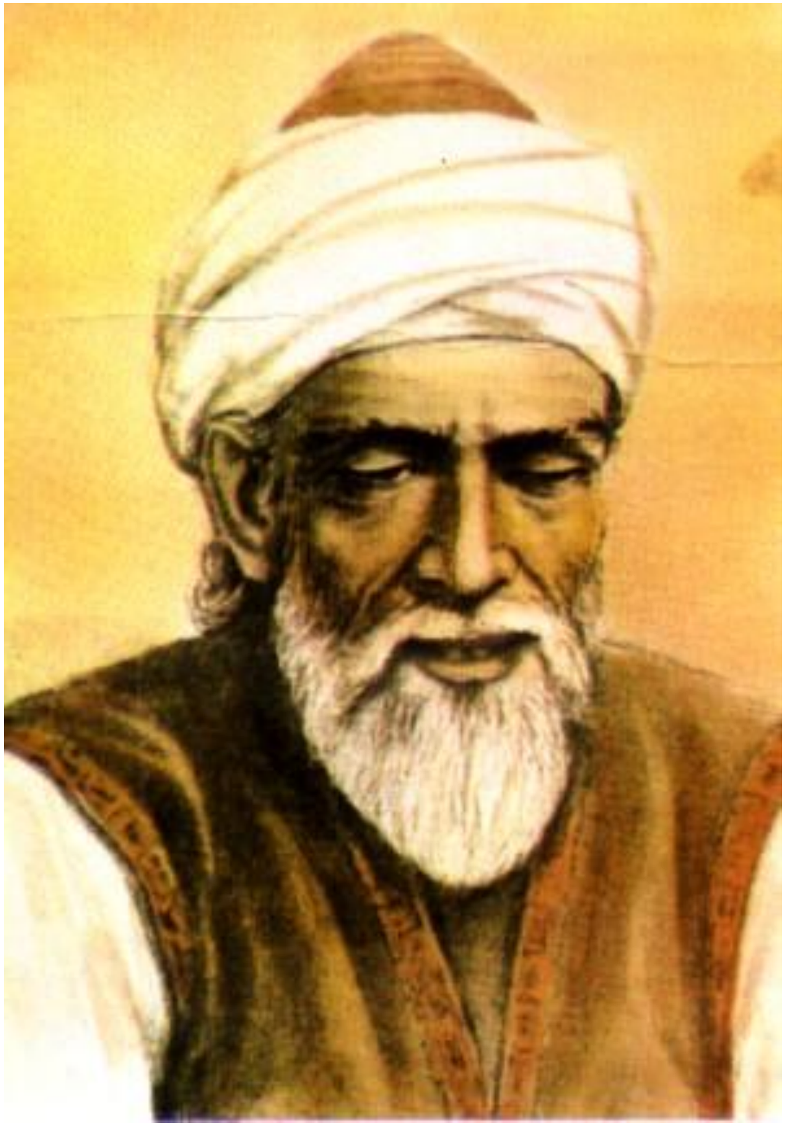


Scholars at an Abbasid library. Maqamat of al-Hariri Illustration by Yahyá al-Wasiti, 1237

Hulagu Khan's siege of Baghdad (1258)

The House of Wisdom, also known as the Grand Library of Baghdad, refers to either a major public academy and intellectual center in Baghdad or to a large private library belonging to the Abbasid Caliphs during the Islamic Golden Age (754/775 – 1258)

Abu'l Wafa (AD 940–998) (also known as Abu al-Wafa and Abu al-Wafa Buzhjani) was a Persian mathematician and astronomer who worked in Baghdad. He was the first to describe geometrical constructions possible only with a straightedge and a fixed compass, later dubbed a 'rusty compass,' that never alters its radius. He pioneered the use of the tangent function, apparently discovered the secant and cosecant functions, and compiled tables of sines and tangents at 15 arcminute intervals – work done as part of an investigation into the orbit of the Moon. Abu Al-Wafa' was the first to build a wall quadrant to observe the sky.



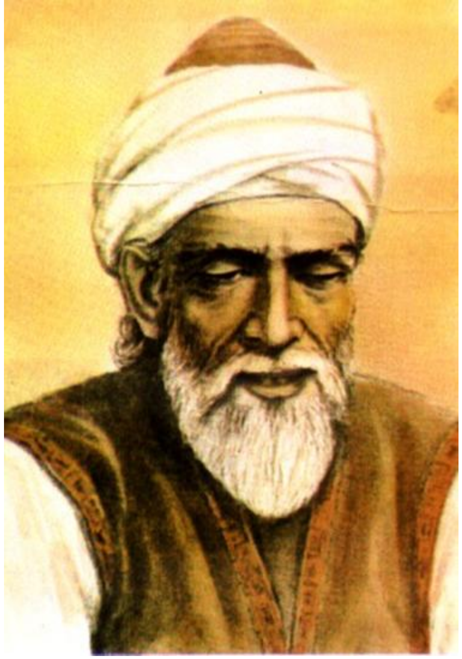
Almagest Abū'l-Wafāʾs large astronomical work, al-majisṭī, or Kitāb al-kāmil ("Complete Book"), closely follows Ptolemy's Almagest.

A book of zij called Zīj al-wāḍiḥ (زيج الواضح), no longer extant.[6]

"A Book on What Is Necessary from the Science of Arithmetic for Scribes and Businessmen" (961-976), (كتاب في ما يحتاج إليه الكتاب والعمال من علم الحساب Kitāb fī mā yaḥtāj ilayh al-kuttāb wa'l-ʿummāl min ʾilm al-ḥisāb).[9] This is the first textbook on practical arithmetic where negative numbers have been used in the medieval Islamic texts.

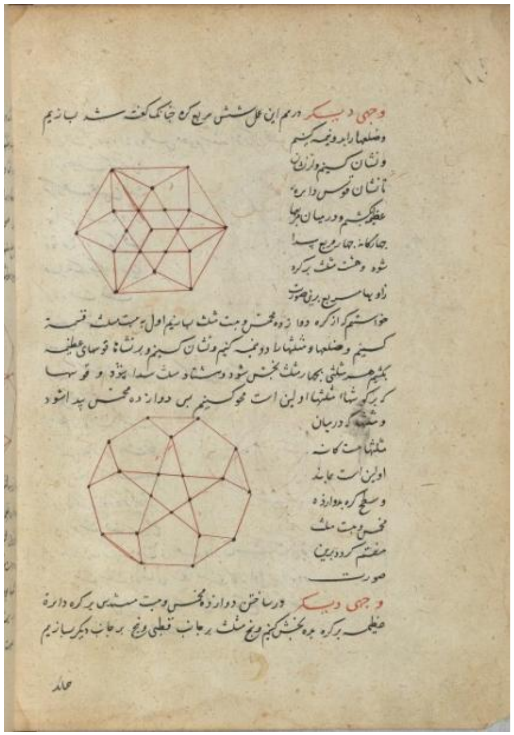
"A Book on Those Geometric Constructions Which Are Necessary for a Craftsman/Artisan" (after990), (كتاب في ما يحتاج إليه الصانع من الأعمال الهندسية Kitāb fī mā yaḥtāj ilayh al-ṣāniʿ min al-aʿmāl al-handasiyya).

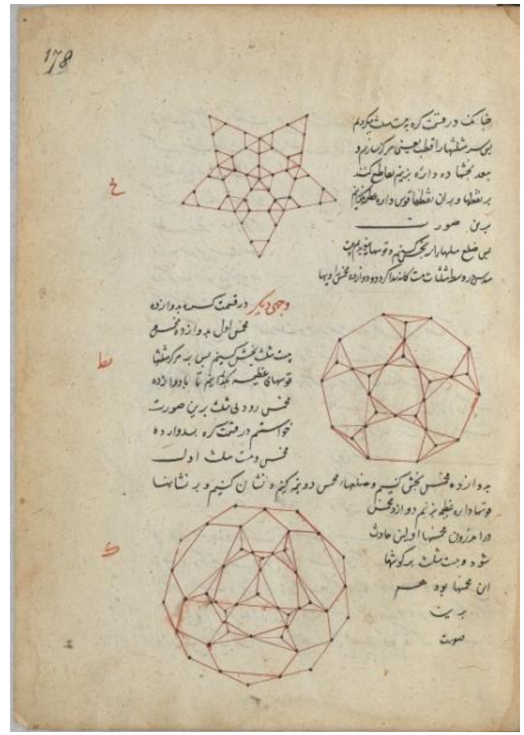
He also wrote translations and commentaries on the algebraic works of Diophantus, al-Khwārizmī, and Euclid's Elements.



Abu ‘l-Wafa described five Archimedean bodies (truncated tetrahedron, truncated icosahedron, truncated octahedron, cuboctahedron, icosidodecahedron) and suggested their construction diagrams as well as almost perfect perspectival drawings for cuboctahedron and icosidodecahedron. Abu’l-Wafa was probably familiar with at least parts of Pappus’ Collection [1].

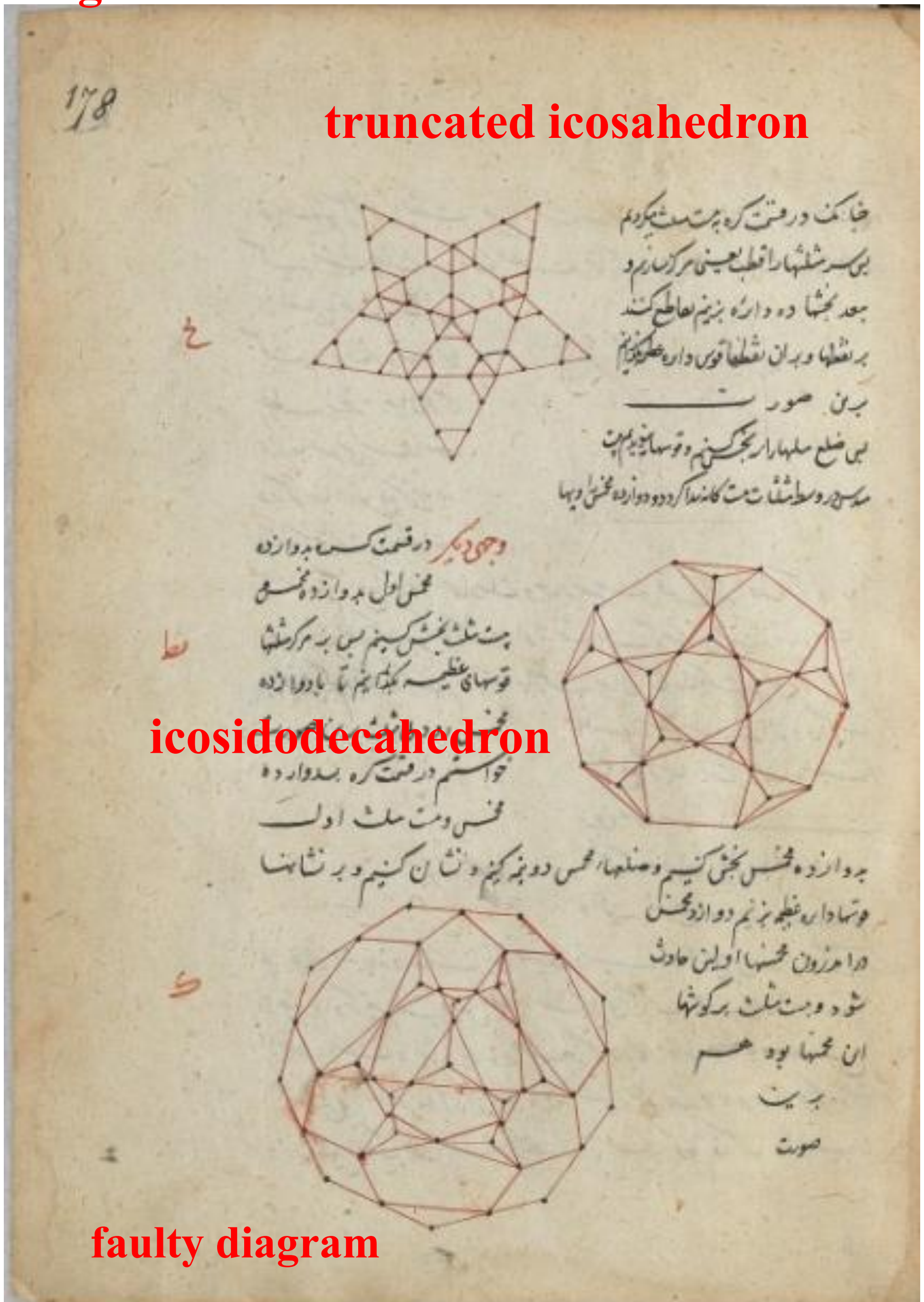
1. Jos Janssen. On the History of Archimedean Bodies and Star Polyhedra during the Renaissance. *Unpublished.*



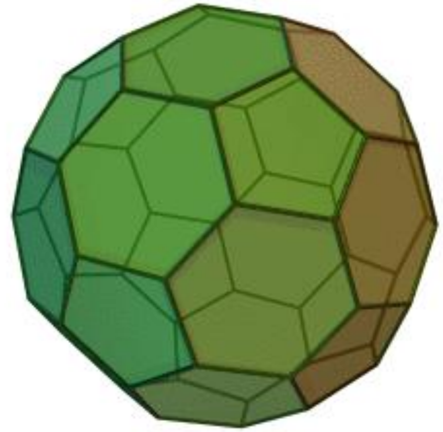


No original Abu’l-Wafa’s manuscript survived. Reprodued pages are from the “Paris manuscript” which is a transcript from the first half of the 17th century, based on a translation of ~1450, in its turn based upon an early 11th century translation. There is no evidence that the drawings in the Paris manuscript are unchanged, compared to the original.

Page f178r

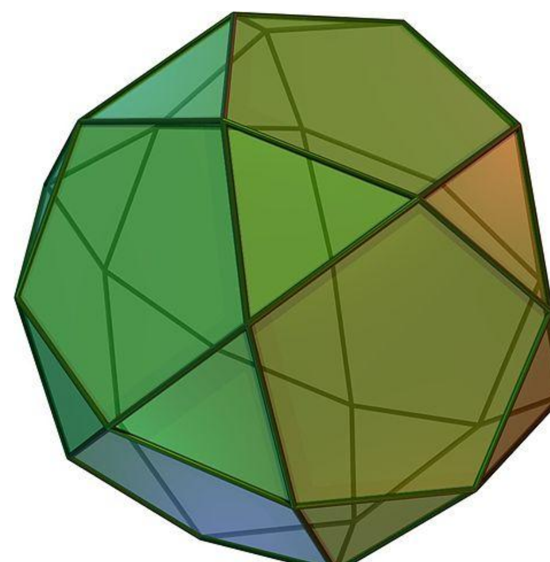


$20_6 + 12_5 = 32$



Only a half of the truncated icosahedron is shown (in a way similar to so-called “Schlegel diagram”

$20_3 + 12_5 = 32$



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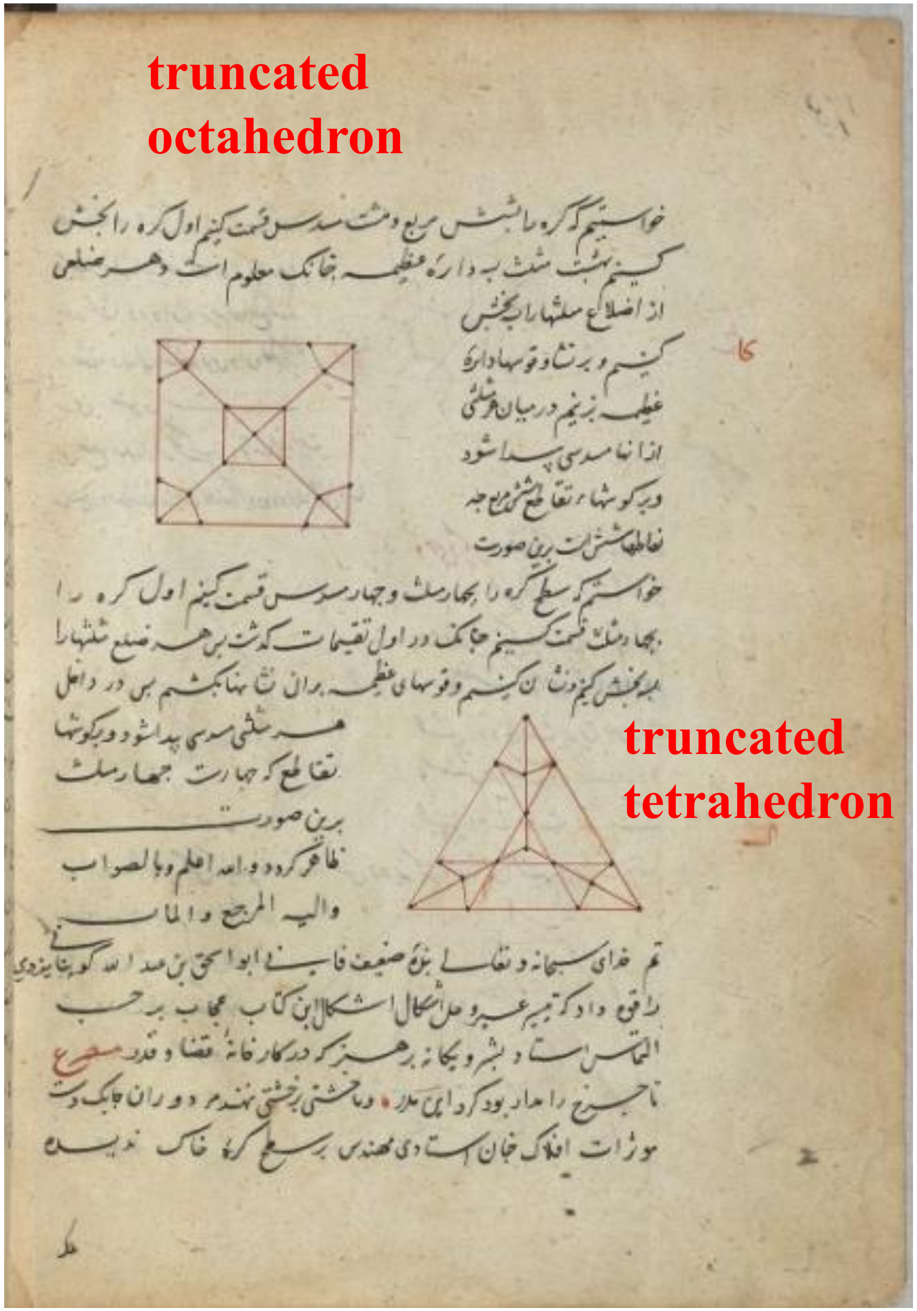
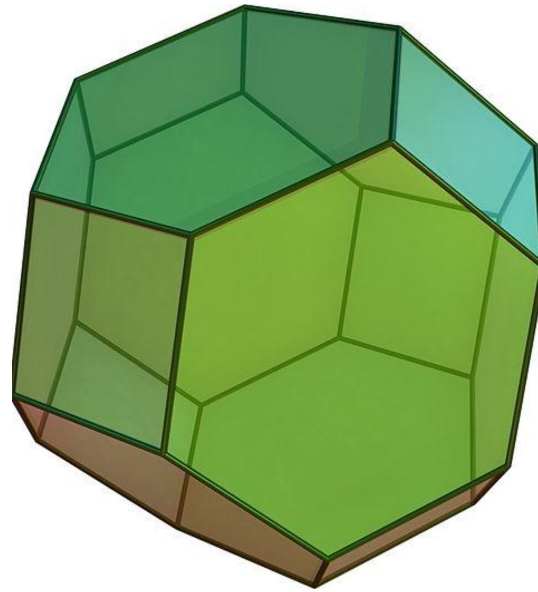
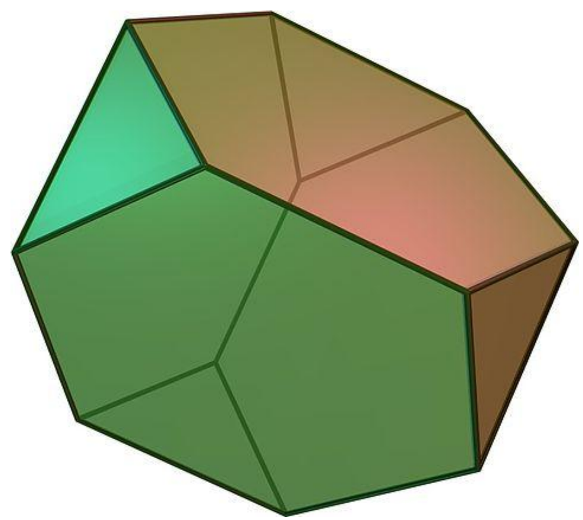




$8_6 + 6_4 = 14$



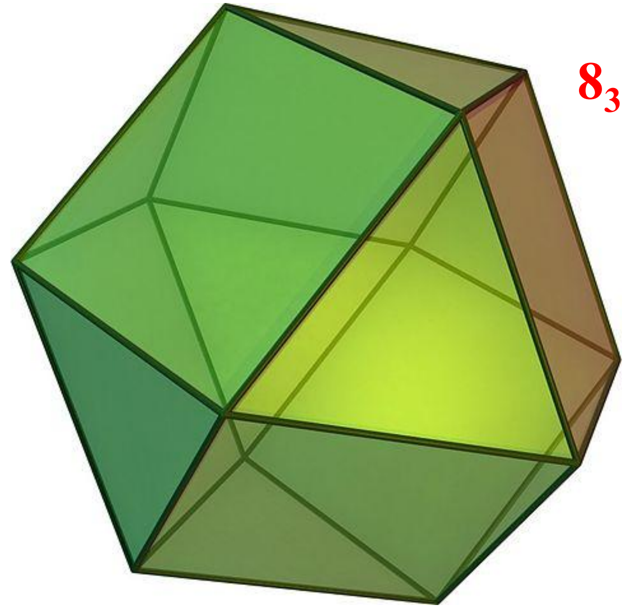
$4_6 + 4_3 = 8$

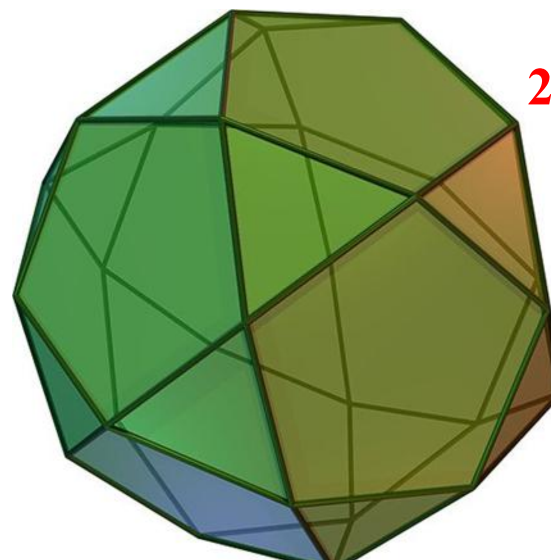
Page f177v

وجهی دیگر در رسم این عمل شش مربع که چنانکه گفته شد بسازیم
وضلعها را بدو نیمه کنیم
ونشان کنیم واز نشان
تا نشان قوس دایره
عظیمه بکشیم ودرمیان هرجا
چهارگانه چهار مربع پیدا
شود وهشت مثلث برکره
زاویها مربع برین صورت

خواستیم که از کره دوازده مخمس وبیست مثلث بسازیم اول بیست مثلث قسمه
کنیم وضلعها و مثلثها را دو نیم کنیم ونشان کنیم وبر نشانها قوسهای عظیمه
بکشیم هر مثلثی بچهار مثلث بخش شود وهشتاد سطح پیدا شود و قوسها
که برگوشها مثلثها اولین است محو کنیم پس دوازده مخمس پیدا شود
ومثلثها که درمیان
مثلثهاست که
اولین است بماند
وسطح کره بدوازده
مخمس وبیست مثلث
منقسم گردد برین
صورت

وجهی دیگر در ساختن دوازده مخمس وبیست مسدس بر کره دایره
عظیمه برکره بده بخش کنیم وپنج مثلث بر جانب قطبی وپنج بر جانب دیگر بسازیم





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Archimedean Solids	Number of			Described by
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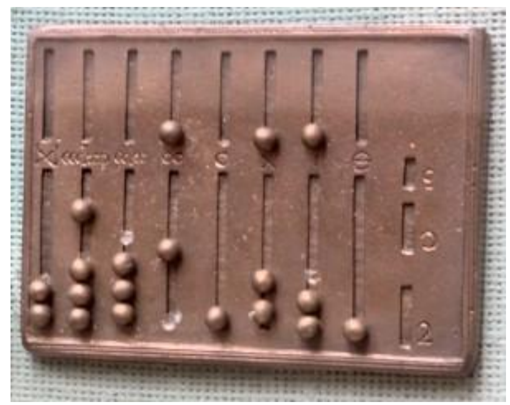
Piero della Francesca (c. 1415 - 1492), originally named Piero di Benedetto, was an Italian painter of the Early Renaissance. To contemporaries he was also known as a mathematician and geometer. Nowadays Piero della Francesca is chiefly appreciated for his art. His painting is characterized by its serene humanism, its use of geometric forms and perspective.

Piero's deep interest in the theoretical study of perspective and his contemplative approach to his paintings are apparent in all his work. In his youth, Piero was trained in mathematics, which most likely was for mercantilism.

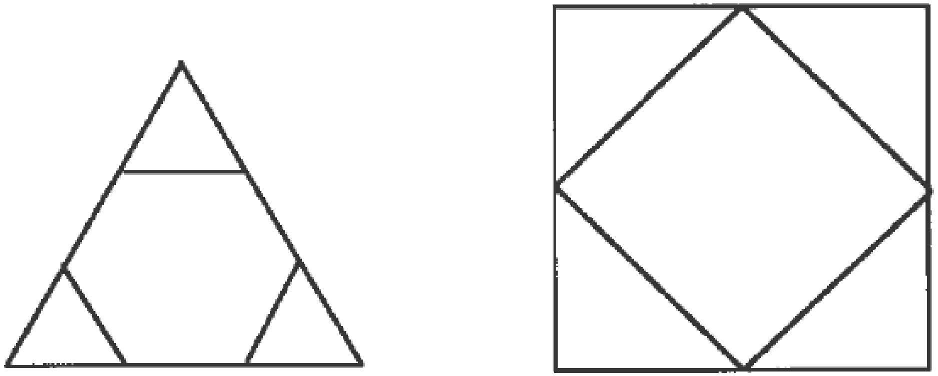
Three treatises written by Piero have survived to the present day.
(1) Trattato d'Abaco (Abacus Treatise) (between 1460 and 1470)

Trattato d'abaco. Autograph manuscript. ?i450s. 21.5 by 14 cm. (Biblioteca Laurenziana, Florence; MS Ashburnham, 359*, fol. 3r)



Roman abacus

Russian «счёты»



a b

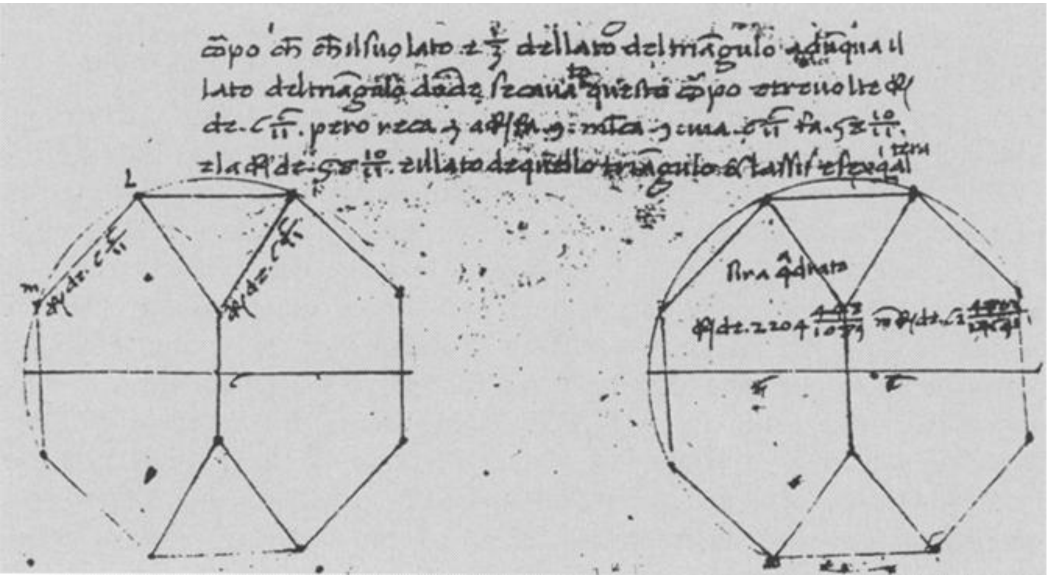
Already in Trattato Piero not only described 5 regular solids but introduced and idea of their truncation for construction of semiregular polyhedra

Fig 1 Truncation of faces of regular polyhedra, to obtain faces of Archimedean polyhedra.

1a Truncation to one third of edge of triangle, used by Piero in his Trattato to obtain the truncated tetrahedron. By the general form of this truncation, a regular n-gon becomes a regular 2n-gon.

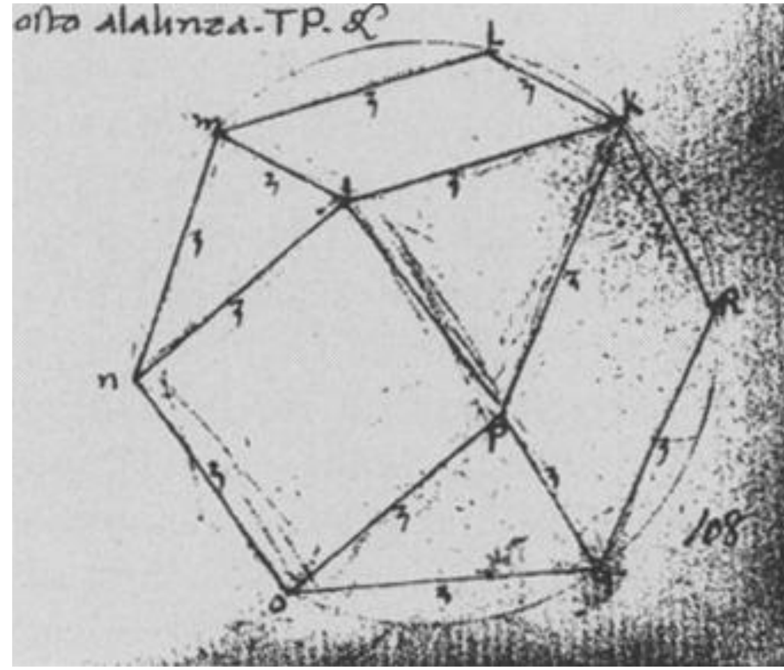
1b Truncation to mid point of edge of square, used by Piero in his Trattato to obtain the cuboctahedron. By this form of truncation, a regular n-gon becomes a smaller regular n-gon.

Reproduced from: J. V. Field. Piero della Francesca and painting as a science.
https://halshs.archives-ouvertes.fr/file/index/docid/52561/filename/05_Field.tif.pdf



Drawings of the truncated tetrahedron, each diameter about 48 mm. From Piero's "Trattato d'abaco".

Diagram of cuboctahedron, diameter about 58 mm. From Piero's "Trattato d'abaco"



Florence, Biblioteca Medicea-Laurenziana, Codice Ashburnhamiano 280 (359*), carta 107 verso. This manuscript is autograph, and it is probable that the illustrations to it were actually drawn by PIERO himself.

Reproduced from: J. V. Field. Rediscovering the Archimedean Polyhedra: Piero della Francesca, Luca Pacioli, Leonardo da Vinci, Albrecht Durer, Daniele Barbaro, and Johannes Kepler. *Archive for History of Exact Sciences*, V. 50, No. 3/4, 241-289 (1997).

Judith Veronica Field. Piero Della Francesca: A Mathematician's Art (Yale University Press, 2005)

De Prospectiva pingendi (On Perspective in painting), between 1474 and 1480.
It is the earliest and only pre-1500 Renaissance treatise solely devoted to the subject of perspective.

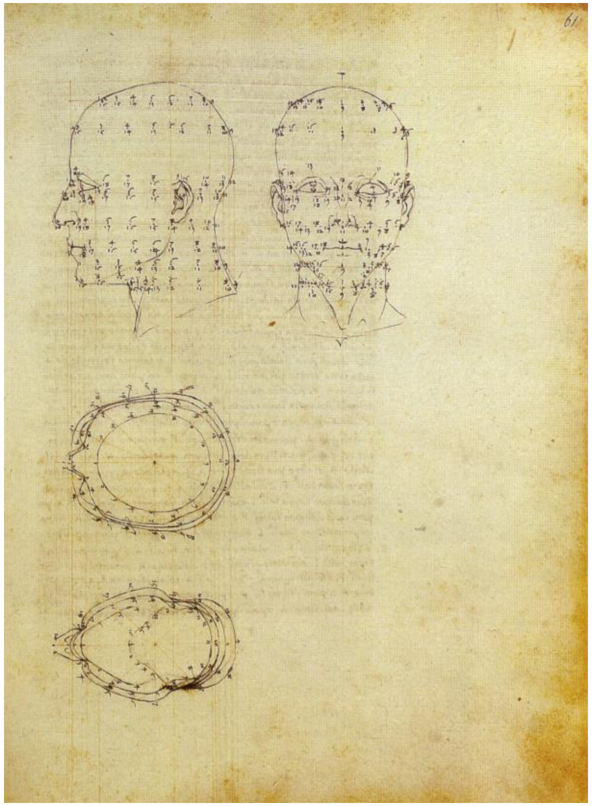
The treatise subjects include arithmetic, algebra, geometry and innovative work in both solid geometry and perspective.

The script consists of three parts:

Part One Disegno, describing techniques for painting faces

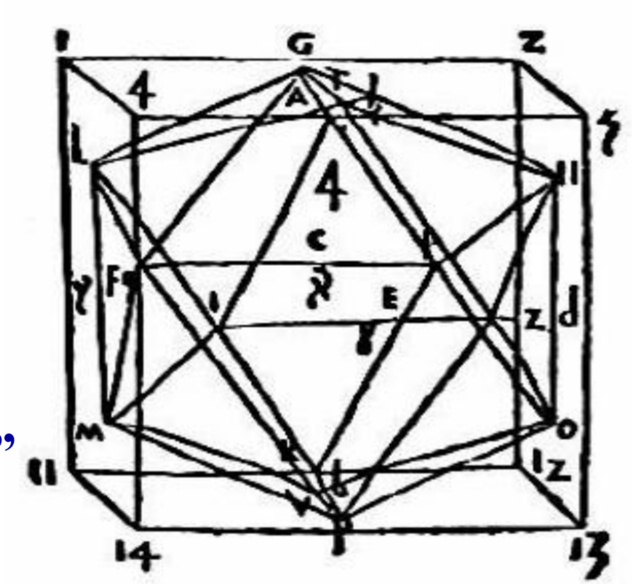
Part Two Commensurazio, describing perspectives

Part Three Coloro, describing techniques for creating perspectives by using colors

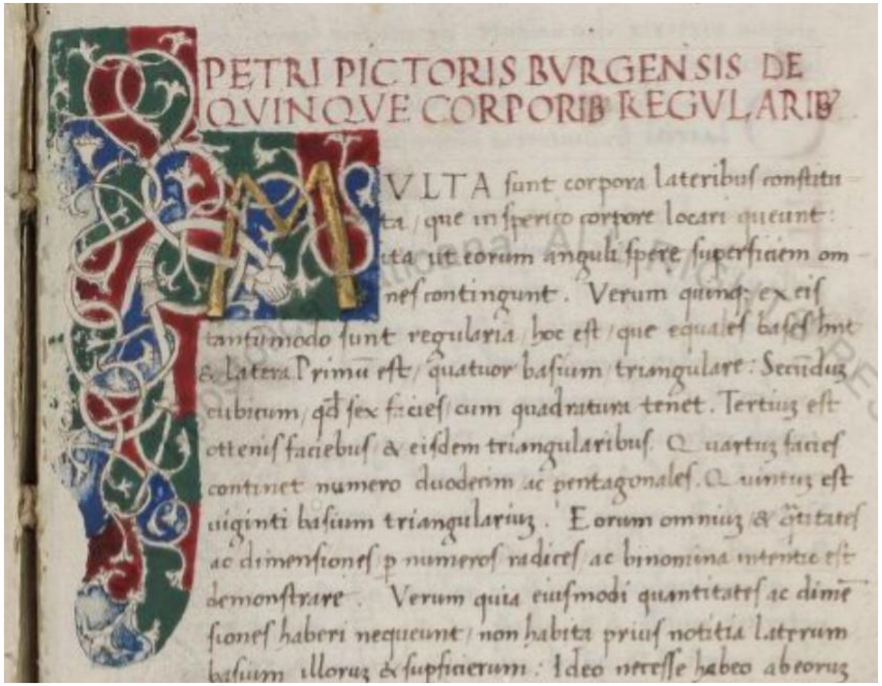


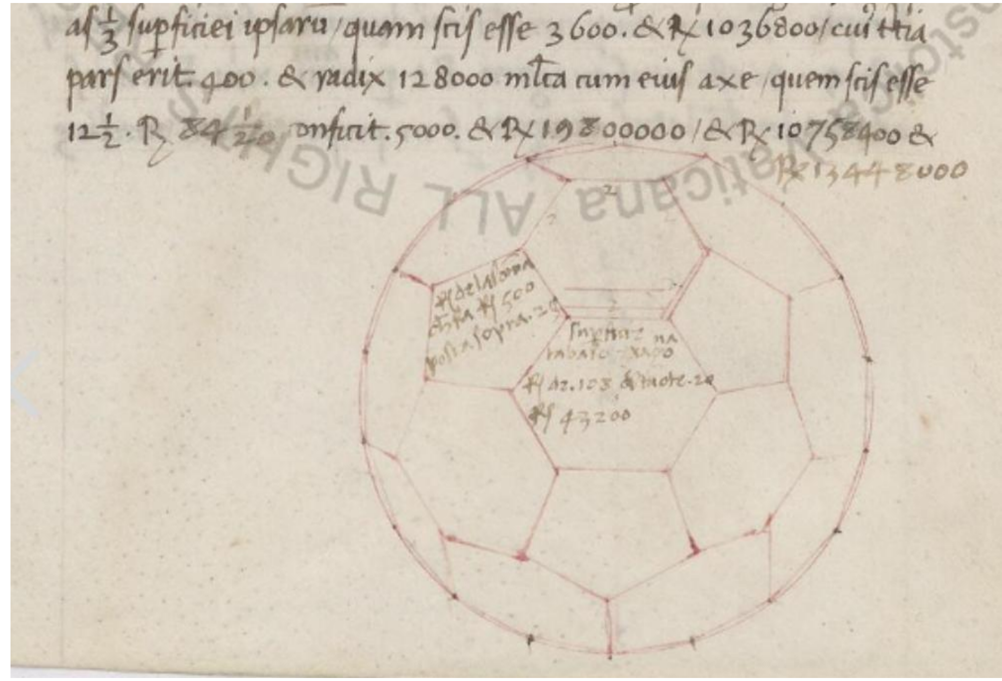
Perspective of anatomy from "*De prospectiva Pingendi*".

An icosahedron in perspective from "De prospectiva pingendi"



Piero della Francesca (1420 - 1492)





The oldest known picture of truncated icosahedron from Piero della Francesca's manuscript "Short book on the five regular solids" ("Libellus de quinque corporibus regularibus"), dated 1480 or early 1490s, Vatican Library.

Along with the Platonic solids, it includes descriptions of five of the thirteen Archimedean solids, and of several other irregular polyhedra coming from architectural applications. It was the first of what would become many books connecting mathematics to art through the construction and perspective drawing of polyhedra.

Prot.: 2019/1423/S-AA03
Vatican City, October 14th 2019

Dear Professor Katz,

in replay to your kind request, in which you asked us the access to the Vatican Library for your research on Luca Pacioli and Piero della Francesca, we wish to inform you that we are pleased to welcome you in the Vatican Library.

In order to access the Library, you are invited to come to our Admissions Office with a copy of this letter and valid identification document, which will be left at the Police Office at the Sant'Anna entrance to the Vatican City State. You will receive a temporary entrance pass and the materials will be made available to you, according to the rules of the Library which can be found on our website (http://www.vaticanlibrary.va).

The Library is open from Monday to Friday from 8:45 to 5:15 on the days indicated on the calendar which can also be found on our website along with the "Rules for Readers", our catalogue of printed books, the hours and other practical information.

Kind regards,

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Prof. Eugene KATZ
keugene@bgu.ac.il

Autopsy (Greek αὐτός "self" + ὄψις "sight") - pathological or forensic procedure, postmortem examination of the body and internal organs

Angelo Tricca (1817-1884). " Piero della Francesca teaches geometry to Luca Pacioli"

Characteristics of the Archimedean polyhedra

Archimedean Solids	Number of			Described by
	Vertices	Faces[a]	Edges	
Truncated tetrahedron	12	$4_3 + 4_6 = 8$	18	Piero della Francesca
Cuboctahedron	12	$8_3 + 6_4 = 14$	24	Piero della Francesca
Truncated cube	24	$8_3 + 6_8 = 14$	36	Piero della Francesca
Truncated octahedron	24	$6_4 + 8_6 = 12$	36	Piero della Francesca
Rhombicuboctahedron	24	$8_3 + 6_4 + 12_4 = 26$	48	Luca Pacioli
Snub cube	24	$24_3 + 8_3 + 6_4 = 38$	60	Albrecht Dürer
Icosidodecahedron	30	$20_3 + 12_5 = 32$	60	Luca Pacioli
Truncated cuboctahedron	48	$12_4 + 8_6 + 6_8 = 26$	72	Albrecht Dürer
Truncated dodecahedron	60	$20_3 + 12_{10} = 32$	90	Piero della Francesca
Truncated icosahedron	60	$12_5 + 20_6 = 32$	90	Piero della Francesca
Rhombicosidodecahedron	60	$20_3 + 30_4 + 12_5 = 62$	120	Luca Pacioli
Snub dodecahedron[b]	60	$60_3 + 20_3 + 12_5 = 92$	150	Johannes Kepler
Truncated icosidodecahedron	120	$30_4 + 20_6 + 12_{10} = 62$	180	Daniele Barbaro (1514 - 1570)

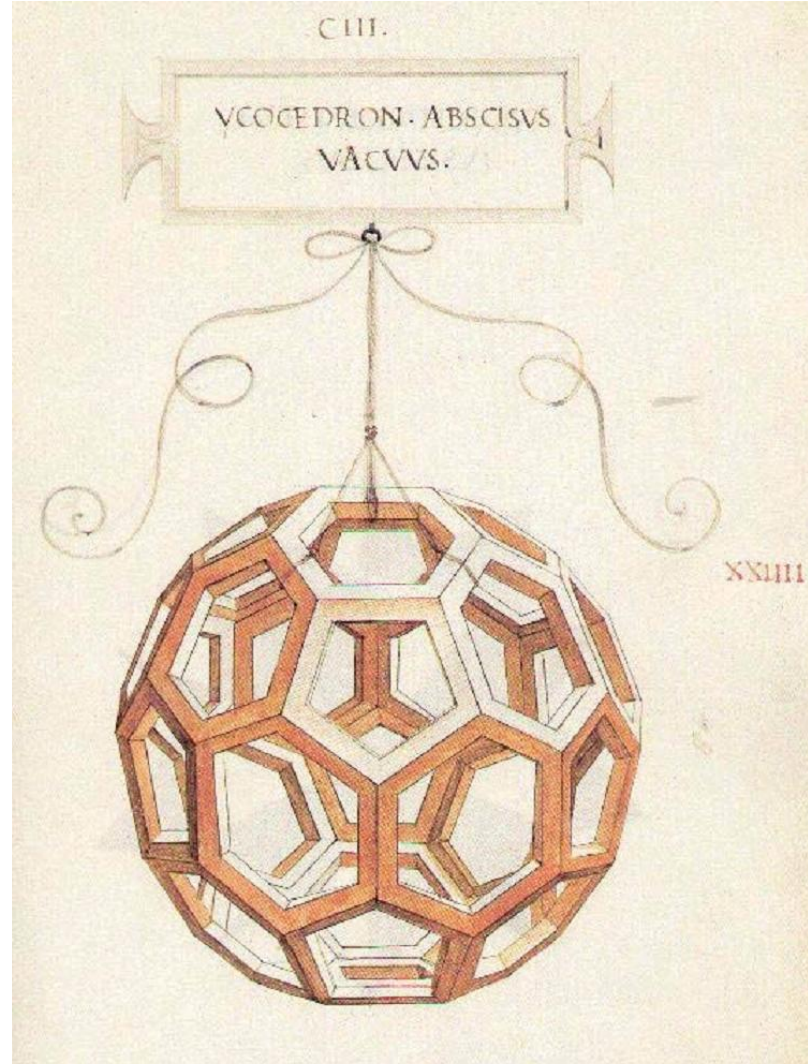
Fra Luca Pacioli

In 1475, he started teaching in Perugia as a private teacher before becoming first chair in mathematics in 1477. During this time, he wrote a comprehensive textbook in the vernacular for his students.

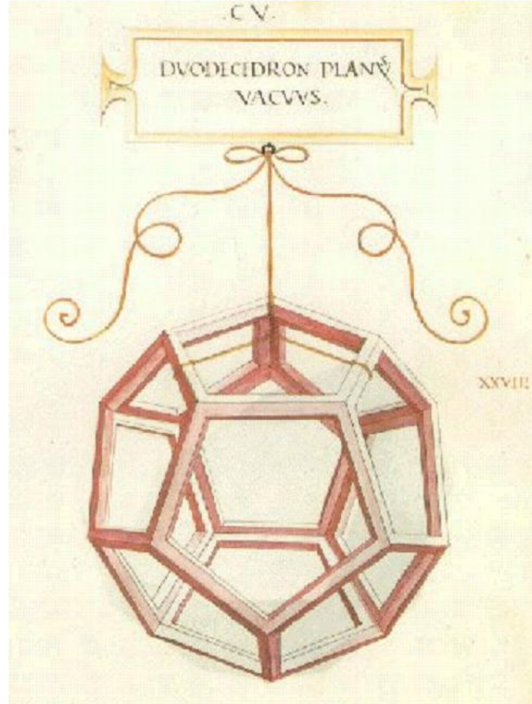
In 1494, his first book, "Summa de arithmetica, geometria, Proportioni et proportionalita", was published in Venice. In 1497, he accepted an invitation from Duke Ludovico Sforza to work in Milan. There he met, taught mathematics to, collaborated, and lived with Leonardo da Vinci. In 1499, Pacioli and Leonardo were forced to flee Milan when Louis XII of France seized the city and drove out their patron.

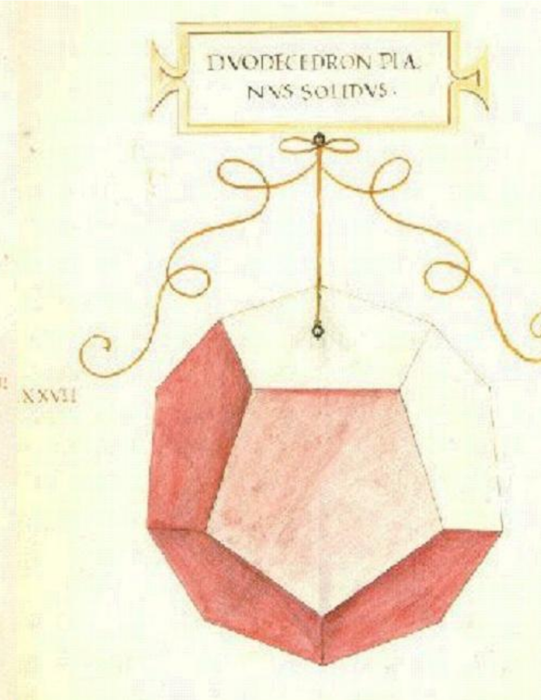
With the fall of Ludovico Pacioli fled Milan together with Leonardo travelling to Florence, by way of Mantua and Venice, where they shared a house. Although both undertook journeys to work in other cities they remained together in Florence until 1506. From 1506 until his death in his hometown in 1517 Pacioli went back to his peripatetic life as a teacher of mathematics. At his death he left behind the unfinished manuscript of a book on recreational mathematics, "De viribus quantitates" ("On the Game of Chess"), which he had compiled together with Leonardo.

“Divina proportione” **(written in Milan in 1496–98, published in Venice in 1509). Two versions of the original manuscript are extant, one in the Biblioteca Ambrosiana in Milan, the other in the Bibliothèque Publique et Universitaire in Geneva. The subject was mathematical and artistic proportion, especially the mathematics of the golden ratio and its application in architecture. Leonardo da Vinci drew the illustrations of the regular and semiregular solids in “Divina proportione” while he lived with and took mathematics lessons from Pacioli. Leonardo's drawings are probably the first illustrations of skeletal solids, which allowed an easy distinction between front and back. The work also discusses the use of perspective by painters such as Piero della Francesca, Melozzo da Forlì, and Marco Palmezzano**



The book contains 60 drawings of polyhedra attributed to Leonardo da Vinci: prisms, pyramids, cylinders, spheres, the five Platonic solids and six of the thirteen Archimedean solids.



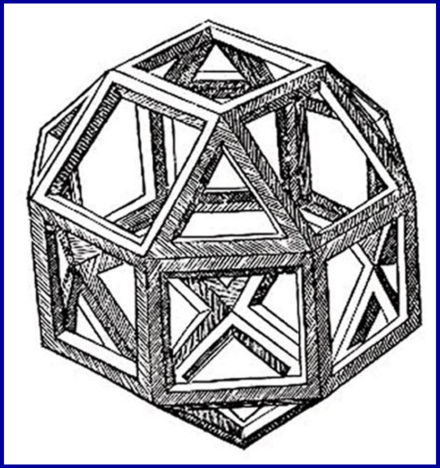


Pacioli often describes four versions of the solids: hollow and solid (i.e. polyhedra in skeletal form, consisting of a hollow structure in which almost all edges are visible versus non-transparent polyhedra with opaque faces) and with or without pyramids on each face (i.e. "elevated" or "plane" polyhedra).
Pacioli described the 5 regular Platonic polyhedra and their "dependants", by which Pacioli meant: polyhedra obtained out of the regular bodies by respectively subtraction of pyramids (truncation, leading to Archimedean bodies), and addition of pyramids (elevation or augmentation, leading to star polyhedra). Leonardo's spectacular illustrations, particularly of the star polyhedra led to the massive popularity of the printed version of the book, especially in the first half of the 16th century.

Luca Pacioli, 1445-1517

Rombicuboctahedron

Jacopo de Barbari (1460 — 1516). Luca Paciolli's portrait, 1495-1500



Rhombicuboctahedron.
(a) Leonardo’s printing in “The Devine Proportion” (1509). (b) National Library of Republic of Belarus in Minsk (2006, architects Mihail Vinogradov and Viktor Kramarenko). Photo by Galya Katz.

Memorial desks of Piero della Francesca and Luca Pacioli in San Sepolcro.

Photos by E. Katz

In honor of Piero della Francesca
in the fifteenth century
the sovereign painter
from whose mastery Perugino
learned the marvels of art
and Italy the geometric principles of perspective
here where her great son lived
and where at 82 years he ascended into the heavens
this memorial is placed in the year 1876
by a grateful and reverential country

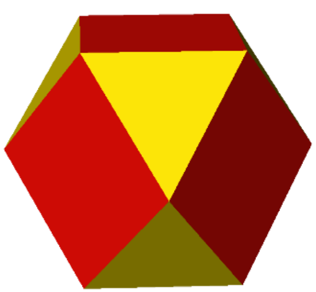
To Luca Pacioli who was friend and consultant
to Leonardo da Vinci
and Leon Battista Alberti
who first gave to algebra the language
and the structure of science
and whose was the great discovery
of applying it to geometry…
The people of San Sepollcro upon
the initiative of the society of workers
ashamed of 320 years of oblivion
to the great fellow citizen
placed [this memorial] in 1878

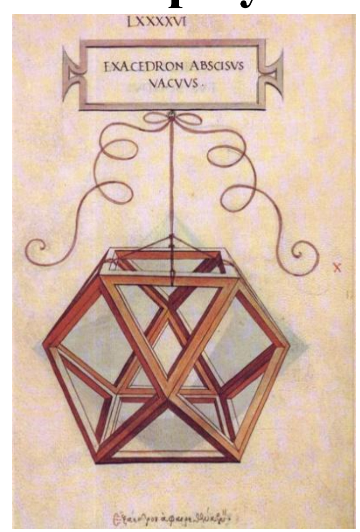
"The Mathematical Intelligencer", 46 (3), 255 - 259 (2024)

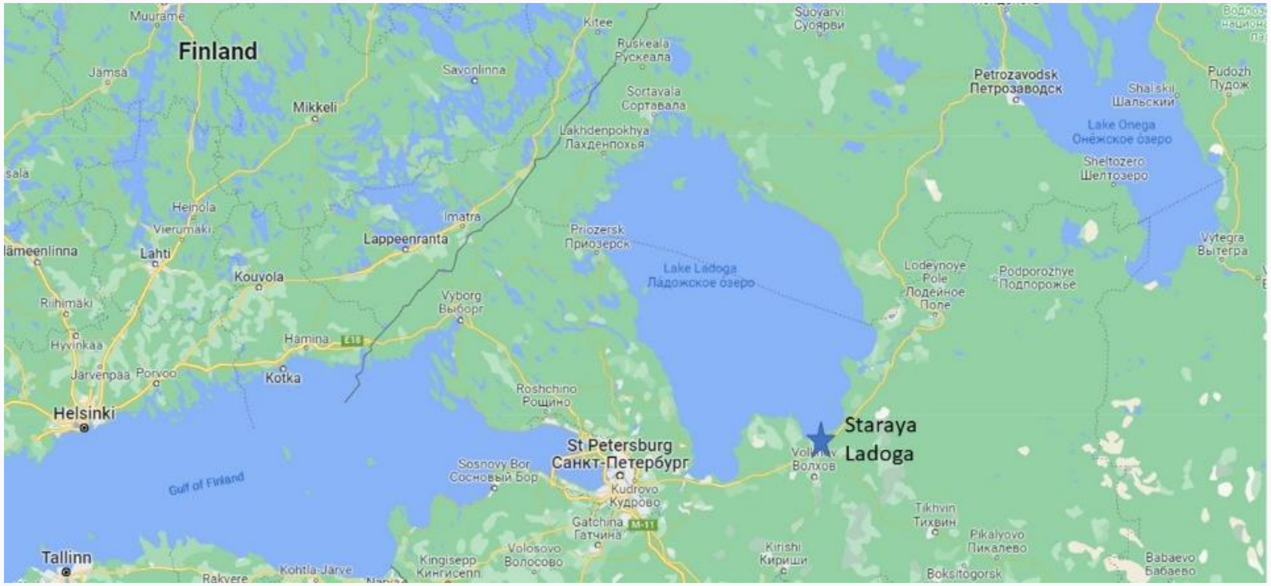
Archimedean cuboctahedron: The Medieval Journey from the Middle East to Northern Russia

Andrey Yu. Chernov, Eugene A. Katz

Abstract. **Bronze cuboctahedral weights dated to the VIII-X centuries were found in northwest Russia near Ladoga, one of the most important trading centers in Eastern Europe in the VIII-X centuries. The history of the mathematical study of cuboctahedron and more generally of the entire family of Archimedean solids in the Middle East and Europe supports the archeological hypothesis about the origin of these artifacts and trading contacts between Europe and the Islamic Caliphate at that time when European mathematicians were not aware of such polyhedra, but Arab-Persian scientists and craftsmen were.**







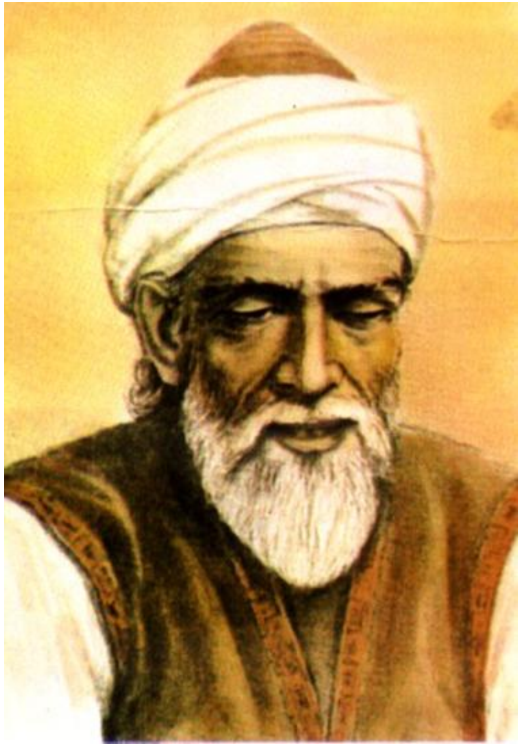
Ladoga is known as the oldest town in Russia.

It is estimated that up to 95% of all Arab dirhams found in Sweden passed through Ladoga. Merchant vessels sailed from the Baltic Sea through Ladoga to Novgorod and then to Constantinople. This route is known as the trade route from the Varangians to the Greeks.

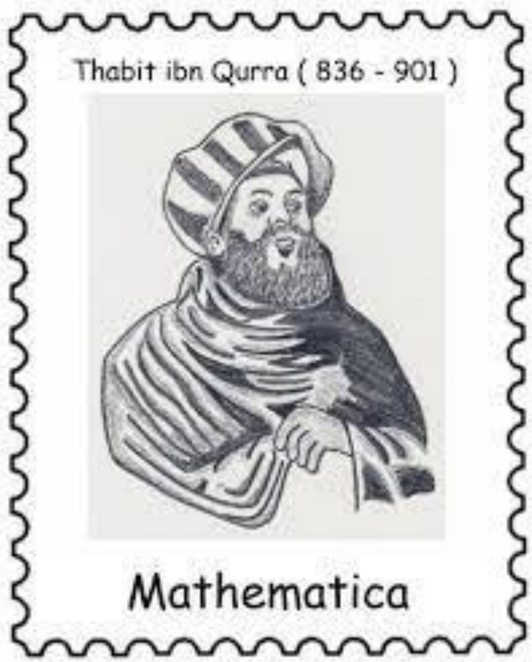
The polyhedral weights found by archeologists in Ladoga, along with dirhams, can be considered as clear evidence of the trade.

You can see the reason for the need to measure the dirham weight at the point of the sale: the coins were simply fragmented by weight. Of course, the scales and weights were subsequently transferred from the Arabs to the Scandinavian merchants.

Abu'l Wafa (AD 940–998). "A Book on Those Geometric Constructions Which Are Necessary for a Craftsman/Artisan" (كتاب في ما يحتاج إليه الصانع من الأعمال الهندسية), after 990).



In fact, the cuboctahedron was described at least a century earlier by Thābit ibn Qurra (826 or 836 – 901), another great mathematician and polymath from the Abbasid Caliphate, in his treatise On the Construction of a Solid Figure with Fourteen Faces Inscribed into a Given Sphere (حول بناء شكل مجسم له أربعة عشر وجهًا منقوشًا في كرة معينة).



In Platonic terms, the cuboctahedron, generated from the octahedron (representing air) and the cube (representing earth), metaphorically represents a phase of a transformational process between earth and sky. This commentary dates to the IX century A.D. by Qusta ibn Luqa, a mathematician, astronomer, and physician. In his treatise "The Introduction to Geometry", he discussed the five regular polyhedra, stating that the ancients compared these polyhedra to the four elements. In this chapter, Qusta ibn Luqa particularly describes a solid with fourteen faces comprising eight equilateral triangles and six equilateral and equiangular quadrilaterals representing "air and earth".

Hogendijk, J. P. 2008. The Introduction to Geometry by Qusta ibn Luqa: Translation and Commentary. Suhayl, 8: 163–221.

Keykavus Mausoleum Sivas, 1220

Karatay Medrese, Konya, 1254

Photos by Andrei Shchetnikov

H. Hisarligil, B. B. Hisarligil. The Geometry of Cuboctahedra in Medieval Art in Anatolia. Nexus Network Journal, 20 (2018), 125–152.

Polyhedral weights in Hecht Museum (Haifa University, Israel)

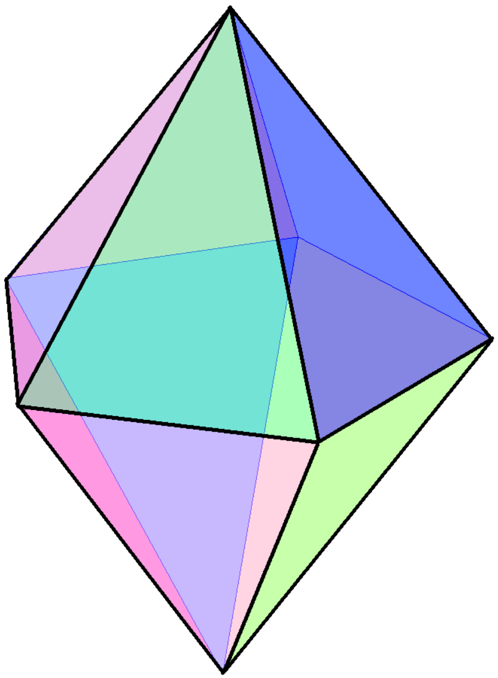
The dirham weights are dated to the Early Islamic Period (VIII–X centuries).
Photo by E. A. Katz.

TRUNCATED ELONGATED OCTAGONAL BIPYRAMID (TEOB)

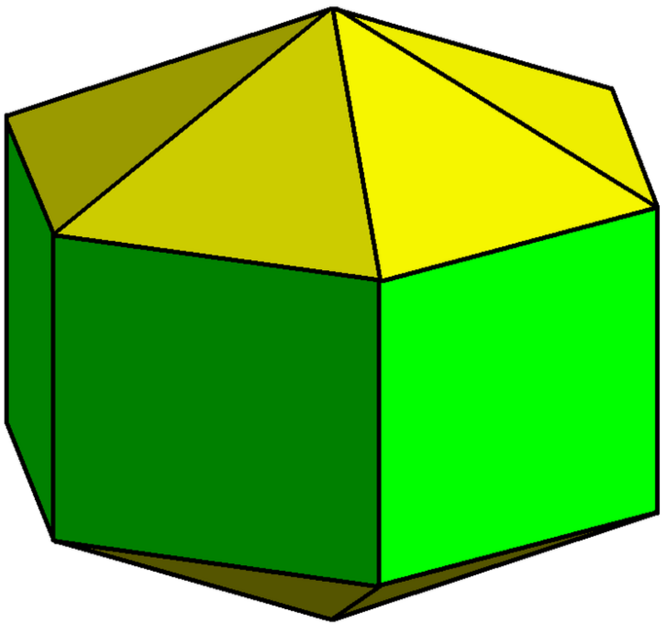
It consists of a central row of eight rectangular (almost square) faces parallel to the vertical eight-fold axis of rotational symmetry, sixteen identical trapezium faces above and below the central row (8+8), and two regular octagonal horizontal faces. It can be approximately inscribed in a sphere with a diameter of 1.5 cm.

E. A. Katz. Journey through time in the footsteps of a polyhedron. Mathematical Intelligencer, v. 47 (2025). https://doi.org/10.1007/s00283-025-10425-x

BIPYRAMID



ELONGATED BIPYRAMID



TRUNCATED ELONGATED BIPYRAMID

TEOB

Mathematicians of the Islamic Caliphate were aware of polyhedra geometry.
To our knowledge, the TEOB was not described by them.

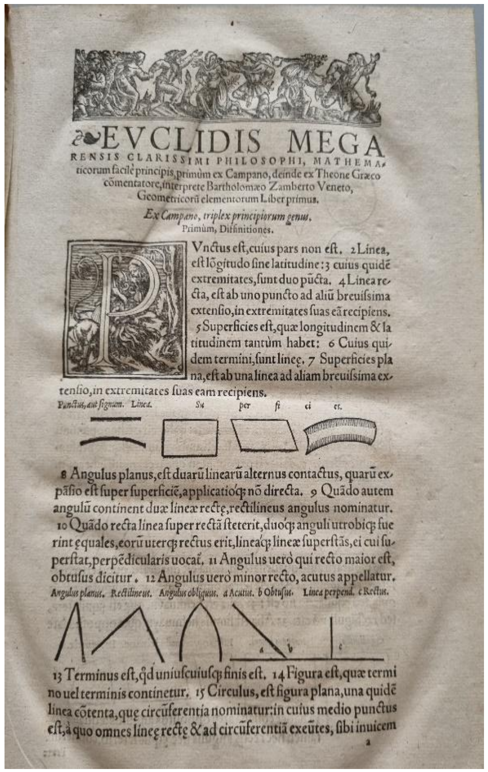
On the one hand, a polyhedron, resembling it, appears in the Early Islamic period via translated versions of Euclid's "Elements." This is the so-called Campanus sphere.

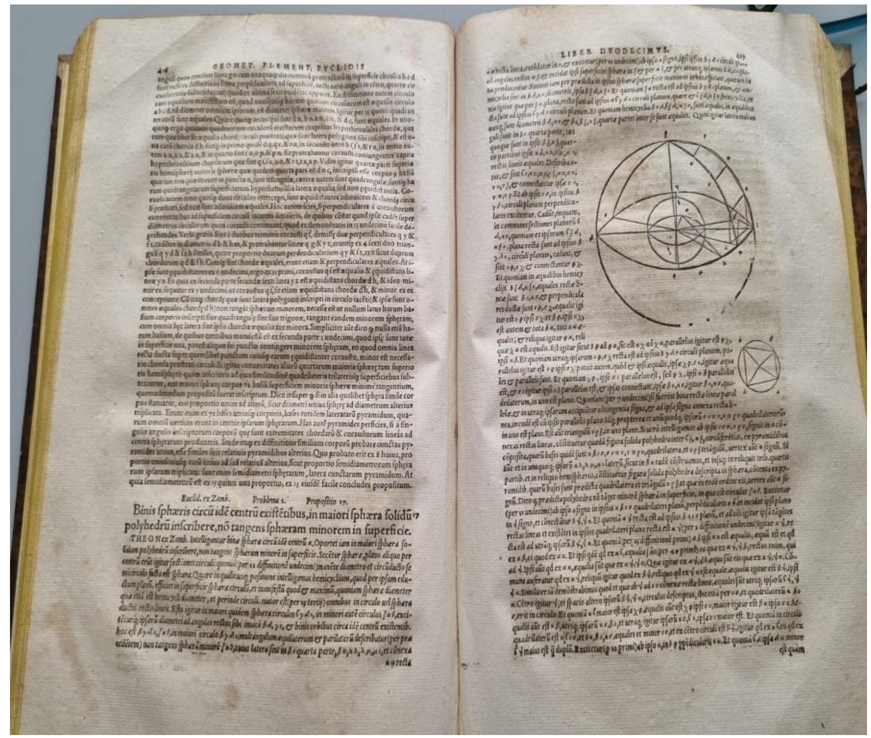
SEPTUAGINTA DUARUM BASIUM SOLIDUM	XXXIX
SEPTUAGINTA DUARUM BASIUM VACUUM	XL

Campanus of Novara (c. 1220, Novara – 1296, Viterbo), Italian mathematician, astronomer, astrologer, and physician. He is mostly known for a popular version of **Euclid's "Elements"** with his own commentaries (1255–1259). Several manuscripts of this treatise survived, and it was reprinted many times in the 15th and 16th centuries. Then Euclid's work was studied in Europe primarily through Campanus' book.

Campanus of Novara. Sculptural portrait by Giuseppe Argenti. This is one of the 48 bas-reliefs of distinguished citizens of Novara (Piedmont, Italy), comprising the unique sculptural pantheon created on the facade of one of the buildings in the city center. Photo by E. A. Katz.

In Proposition 17 of Book XII of "Elements", Euclid suggested how to construct a polyhedron sandwiched between two concentric spheres. Campanus used this construction to describe such a polyhedron with 72 faces, now known as the Campanus sphere.





Euclid's Elementa geometriae. Tr: Adelardus Bathoniensis. Ed: Johannes Campanus. Printed in Vicenza in 1491. Photo by E. A. Katz.

This copy from the Carlo Negroni Public Library in Novara (Italy) is one of many original copies of this incunabula printed in the 15th century to have survived to the present (held in 85 libraries worldwide).

Illustration of the Campanus sphere by Leonardo da Vinci, from Luca Pacioli's "De Divina Proportione."

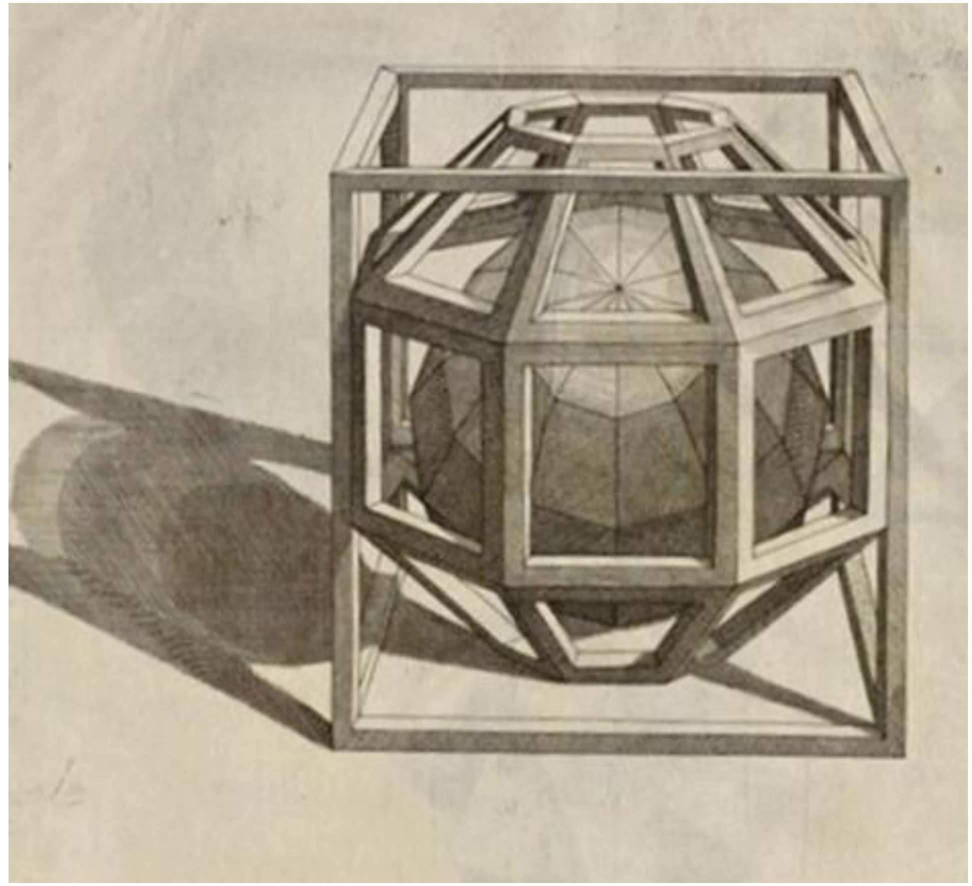
SEPTUAGINTA DUARUM BASIUM SOLIDUM	XXXIX
SEPTUAGINTA DUARUM BASIUM VACUUM	XL

Intarsia by
Fra Giovanni da Verona
in the sacristy of the church Santa Maria in Organo in Verona.

Photo by E. A. Katz.

Lorenzo Sirigatti. «La Pratica di Prospettiva», 1596,Venice.

Reproduced from https://archive.org/details/gri_33125009324340

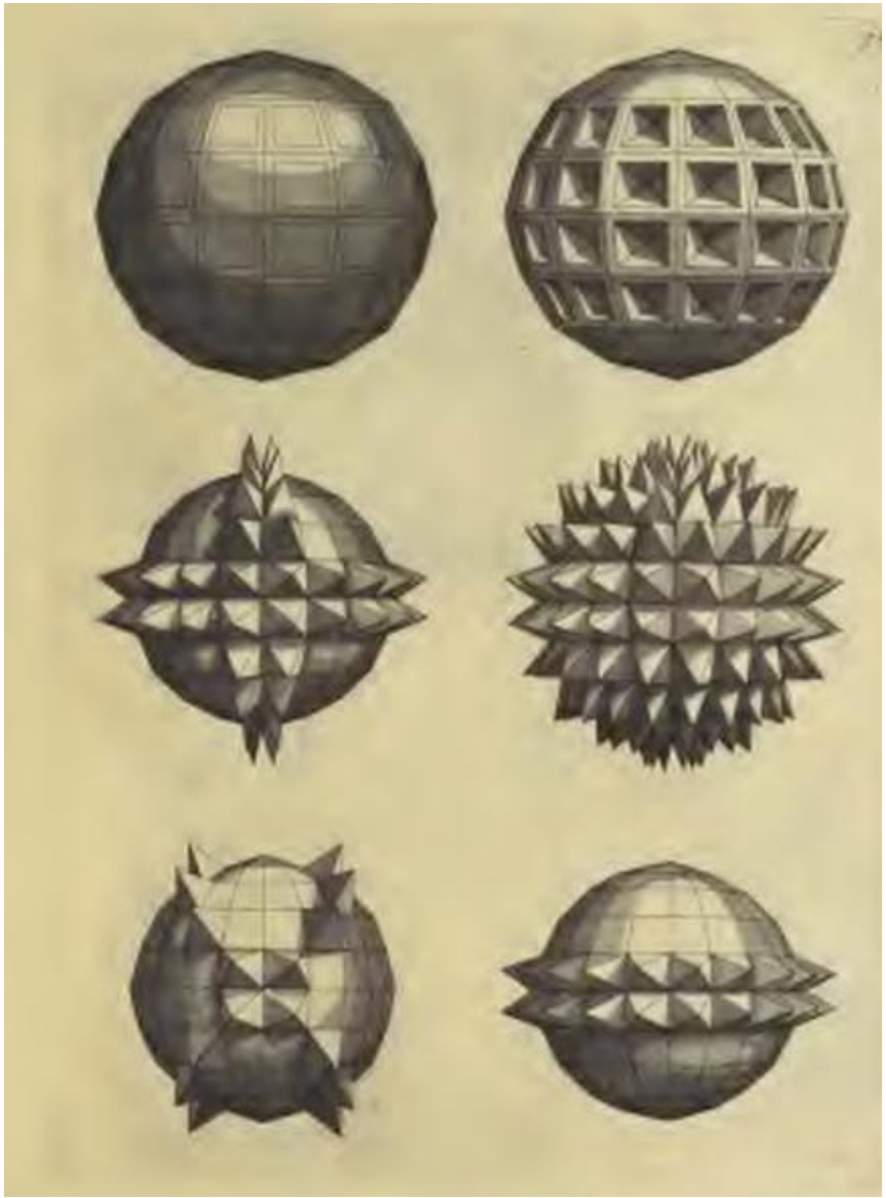


The Campanus sphere is inscribed in the TEOB, which is inscribed in a cube.

Lorenzo Sirigatti (1557- 1625), Florentine architect and perspective artist.

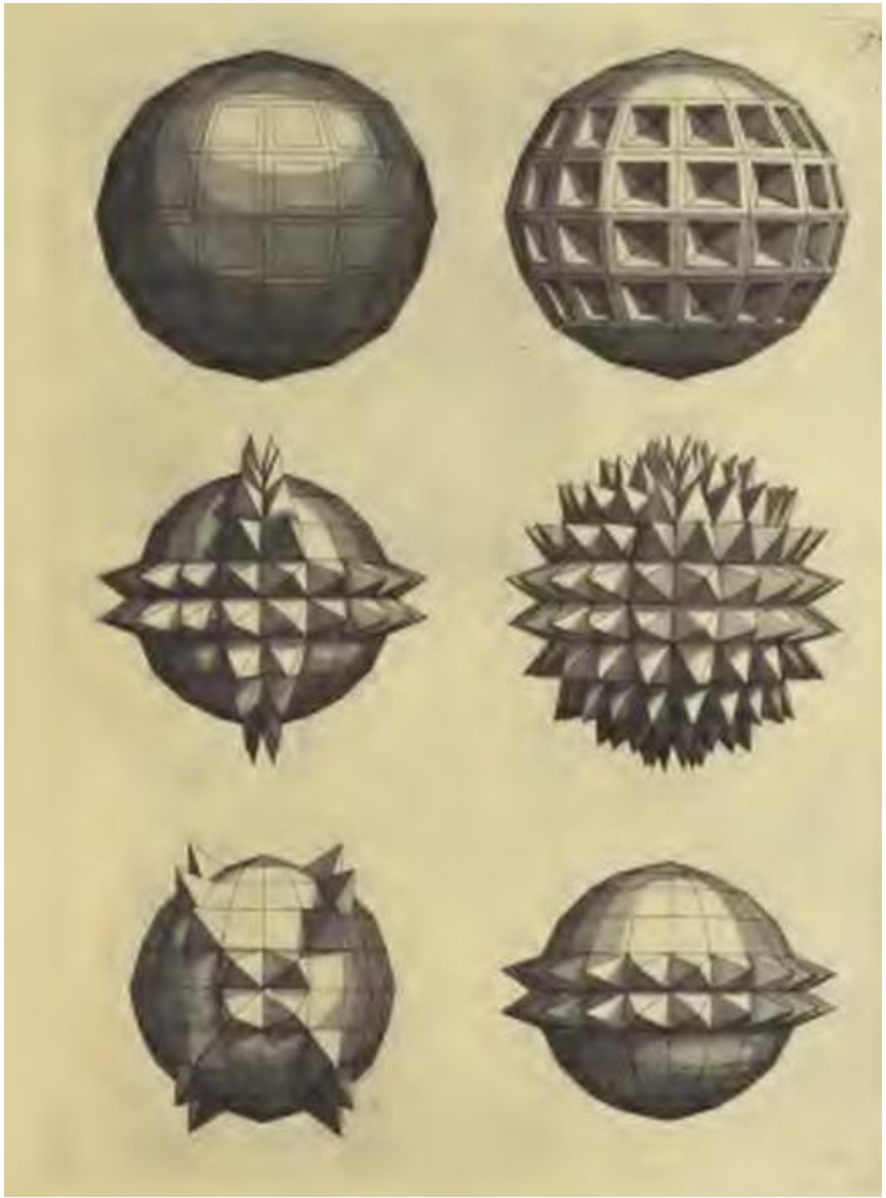
Sirigatti's book features sixty-three engravings based on original drawings. These illustrations depict various polyhedra, including all the Platonic solids, several Archimedean polyhedra, and the Campanus sphere, as well as architectural elements, façades of palaces and churches, and musical instruments.

The engravings are of excellent quality and clarity, and the concise descriptions make the treatise a practical manual that has been widely used by generations of artists and architects in a study of *chiaroscuro* (a technique for representing three-dimensional subjects in the plane through the use of light/shadow patterns of complex geometric forms).



Most famous among those who were influenced by Sirigatti's book's lessons on chiaroscuro was … Galileo Galilei.

He learned to show a pattern of light and shadow from the spikes on a ring diagram. Each spike must cast an appropriate shadow, not so unlike the patches Galileo would later observe using his telescope and interpret as the shadows of mountains protruding up from the surface of the Moon. Indeed, when Galileo and English astronomer Thomas Harriot (c. 1560–621) almost simultaneously pioneered the use of the telescope to study the Moon's surface, it was Galileo's training in chiaroscuro that led him to see mountains and craters where Harriot only saw "strange spottiness".



Galileo Galilei. Wash drawing for illustration of the Moon surface in ‘Sidereus Nuncius’ (1610).

Photo made by E. A. Katz in March 2024 at the exhibition “Celestial splendors. Observing the Sky from Galileo to Gravitational Waves” in Monumental Complex of Santa Maria Novella, Florence.

Palazzo Salvetti Sebregondi, Florence, via Ghibellina, 81. Although having more ancient origins, the palace currently presents characteristics based on the project by Lorenzo Sirigatti , who unified and redesigned the pre-existing architecture commissioned by the Della Fonte family. Photo by E. A. Katz

TEOB in modern design

Lamps and lanterns with a cover shaped in the truncated elongated octagonal bipyramid sphere (a-c) or part of it (d).
Photos made by E. A. Katz in:
(a, b) Turino (Palazzo Carignano, masterpiece of the Piedmonts' Baroque designed in 1679 by the great architect and polymath Guarino Guarini), Florence (c), and Bolzano (d).

According to Edoardo Piccoli, an expert in Guarini's architecture, while the technique and decoration point to the late 19th century (or early 20th), "we probably owe this lantern design to an architect emulating Guarini's elaborate style and geometry".

A couple of days after getting the above-cited opinion, I received another letter from Prof. Piccoli about his exciting finding:

"… while browsing my architectural course images, my eye fell on a drawing by Raphael, circa 1520, with the same lantern shape!".

Raphael, Soldier before the cell of St. Peter,
1515-1520,
pen and ink and wash on white paper. Cabinet of Drawings and Prints of the Uffizi Gallery, Florence.

However, some specific help can be found in the paper “Archimedes Salutes Bramante in a Draft for the School of Athens” by Caroline Karpinski.

It is focused on analysing another masterpiece of almost the same time, the chiaroscuro woodcut “Archimedes”. It is attributed to ‘Ugo da Carpi, after Raphael’.

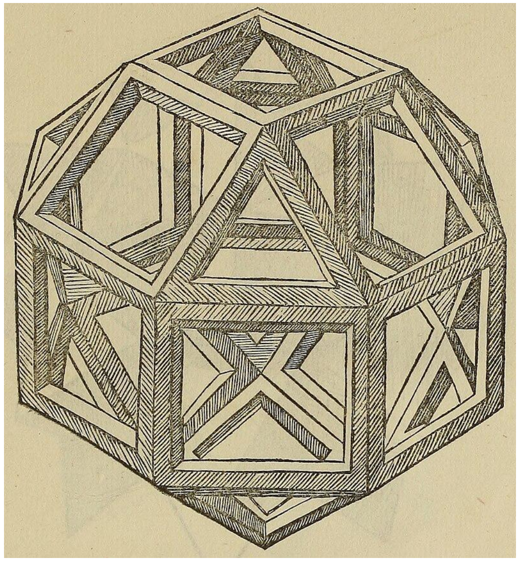
It is clear from the article title that the author believes that Ugo da Carpi reproduced Raphael’s drawing of Archimedes in the engraving, as he was initially supposed to be depicted in “The School of Athens” (1509–1511).

Archimedes (circa 287–212 BCE) has a right to a place among the great thinkers Raphael depicted in the fresco. From a visual point of view, it can be assumed (Joost-Gaugier 2002) that Raphael projected Archimedes to sit in the place of the composition that Diogenes eventually occupies (just below Aristotle). Later, the artist removed this image, leaving only two mathematical protagonists, Pythagoras and Euclid.

Karpinski, C. (2010). Archimedes Salutes Bramante in a Draft for the School of Athens. Artibus et Historiae, Cracow, Iss. 61, 115-132.

Archimedes.
Ugo da Carpi (c. 1450–1480 – c. 1523–1532), after Raphael. 1518-20, Albertina, Vienna.

Rhombicuboctahedron



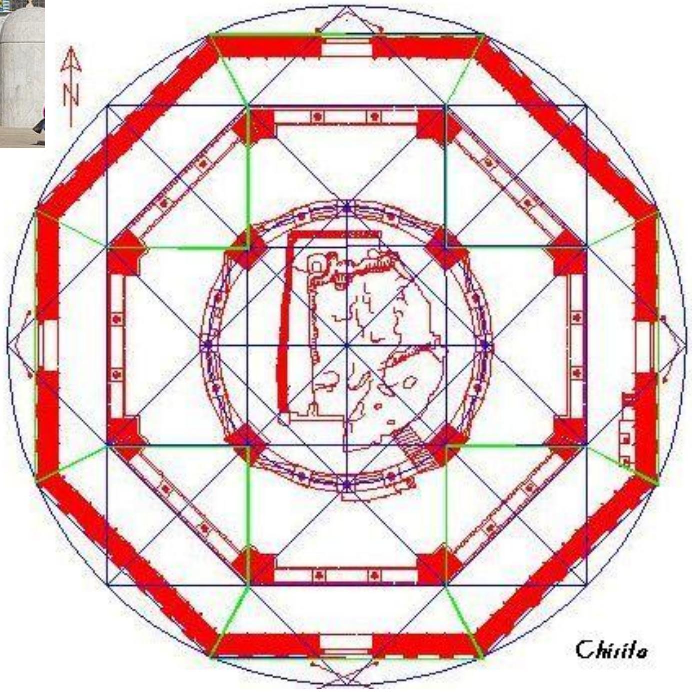
Luca Pacioli in “De Divine Proportione” characterised the rhombicuboctahedron as ‘*architecturally suggestive*’, referring to a building tradition of the octagonal ground plan followed by pagan, Jewish, and Christian sacramental numerology of the number eight.

Luca Pacioli:
«E lorigine de questo fia dalo exacedró vniforme secondo ogni suoi parti tagliato commo similmente alochio la sua material forma cidemostra. E fia la sua sciencia imolte considerationi vtilissima achi bene la accomodare maxime in architectura" (1509, Pars Prima, Ch. LII, f. 15 verso).»

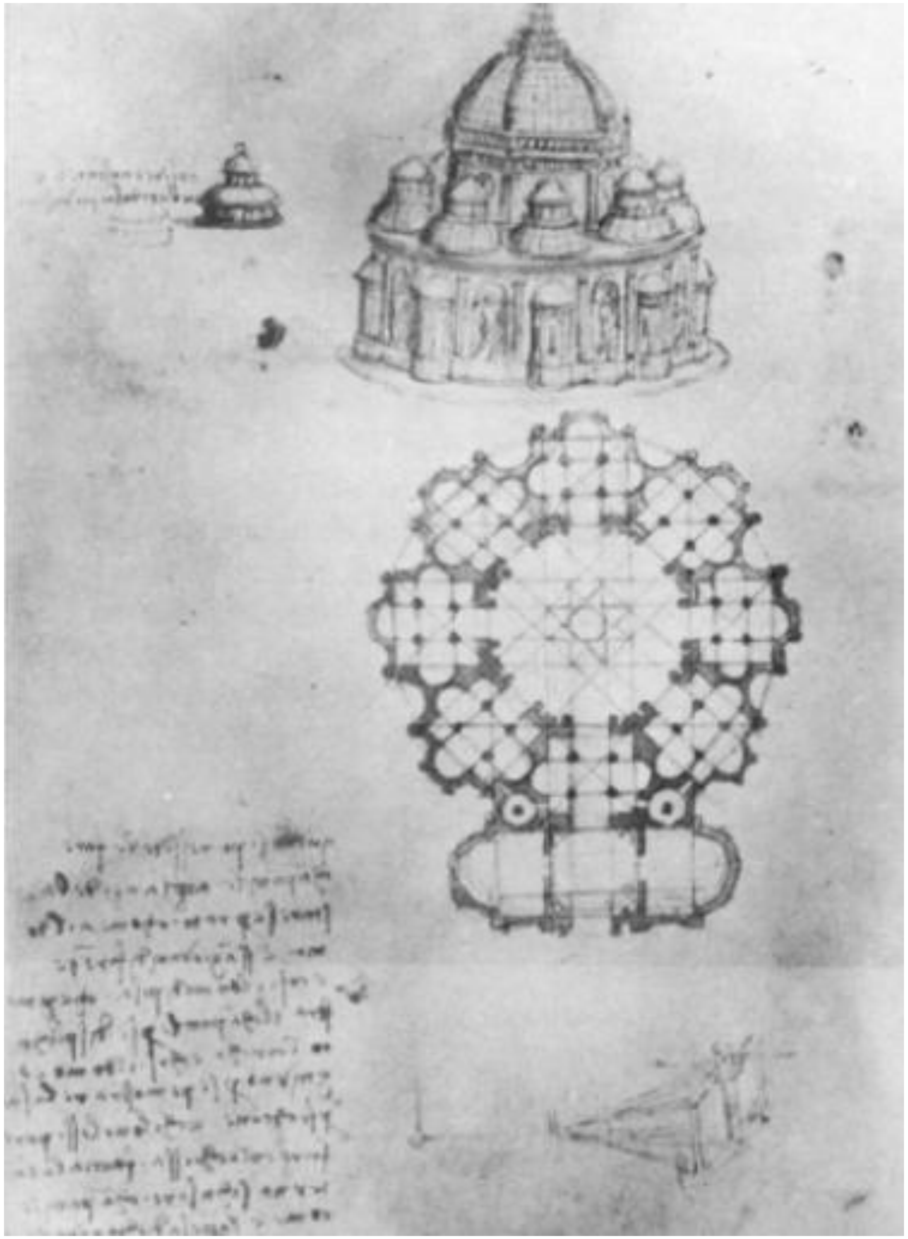
Enlish translation: «And the origin of this will be exaggerated in a form according to each of its parts, cut as if to similarly shape its material form will be demonstrated to us. And its knowledge will be very useful for those who can accommodate it well, especially in architecture».

The oldest surviving octagon-shaped building the Tower of the Winds, Athens, 300 B.C

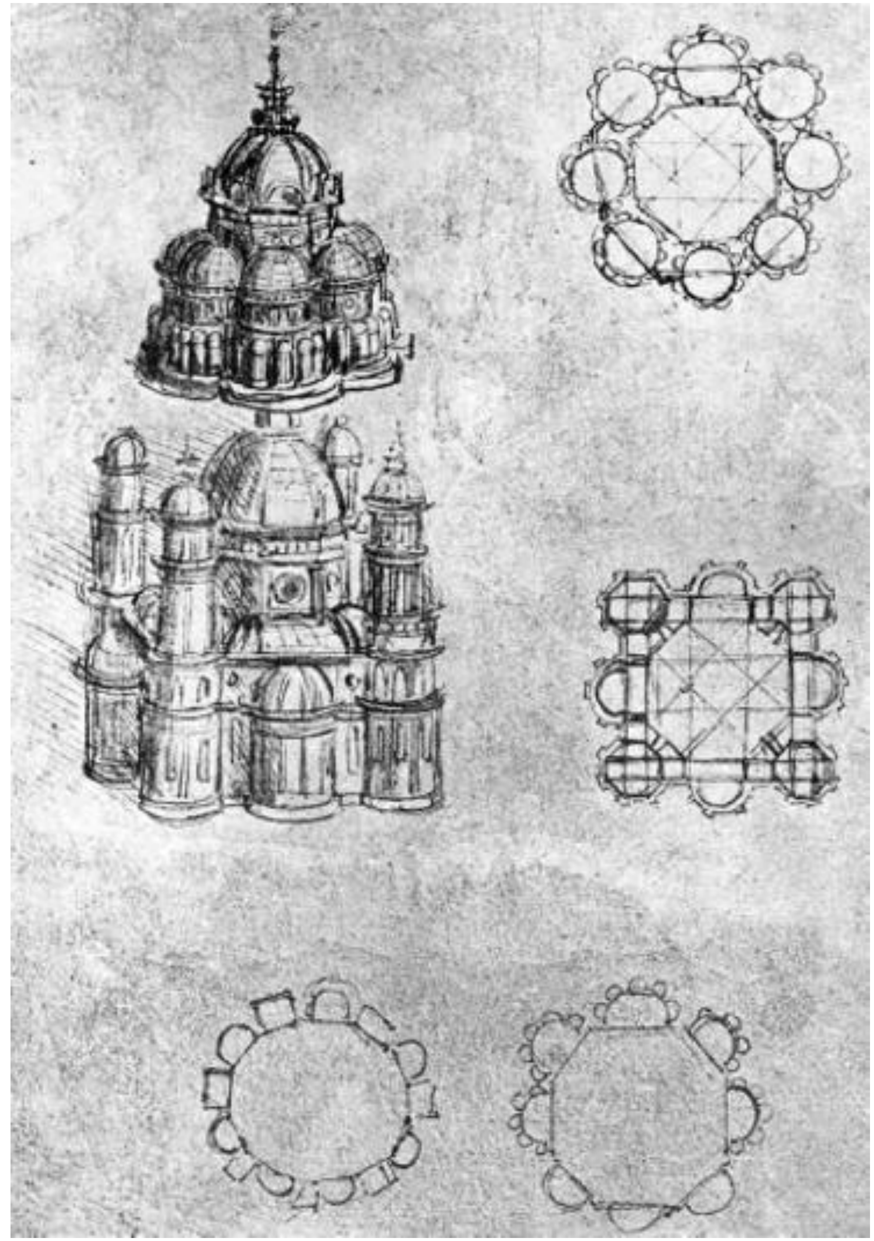
Dome of the Rock, Jerusalem, 687-691 CE



The Florence Baptistery (11th century) and the Florence Cathedral (1436)

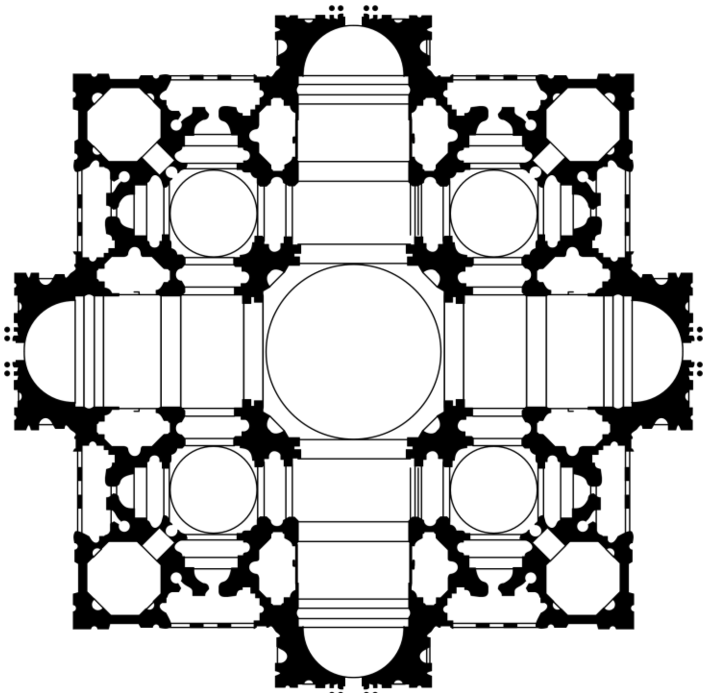


Leonardo da Vinci, plan and bird's-eye view of a centrally-planned church (MS 2307, fol. 5v)

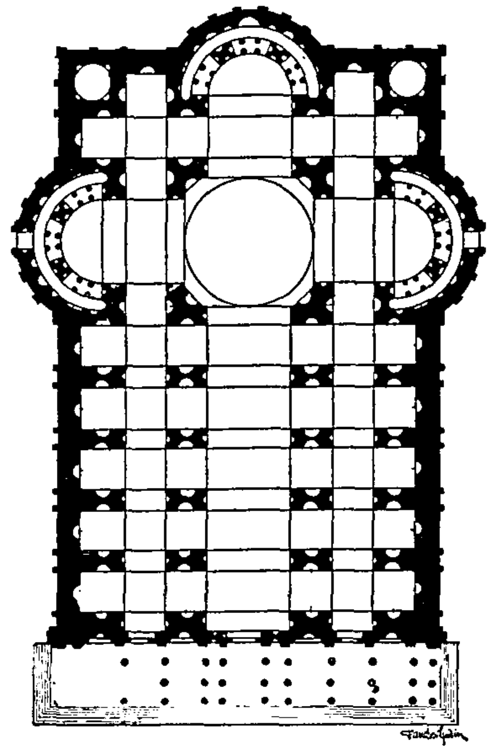


Leonardo da Vinci, drawings of churches. Manuscript B, Institut de France, Paris

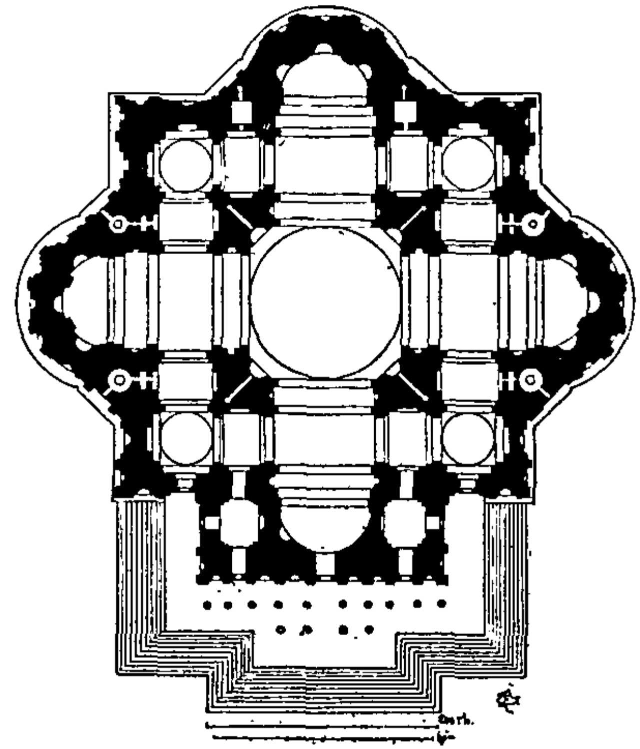
St. Peter's Basilica, Rome, 16th century



Bramante's plan



Raphael's plan

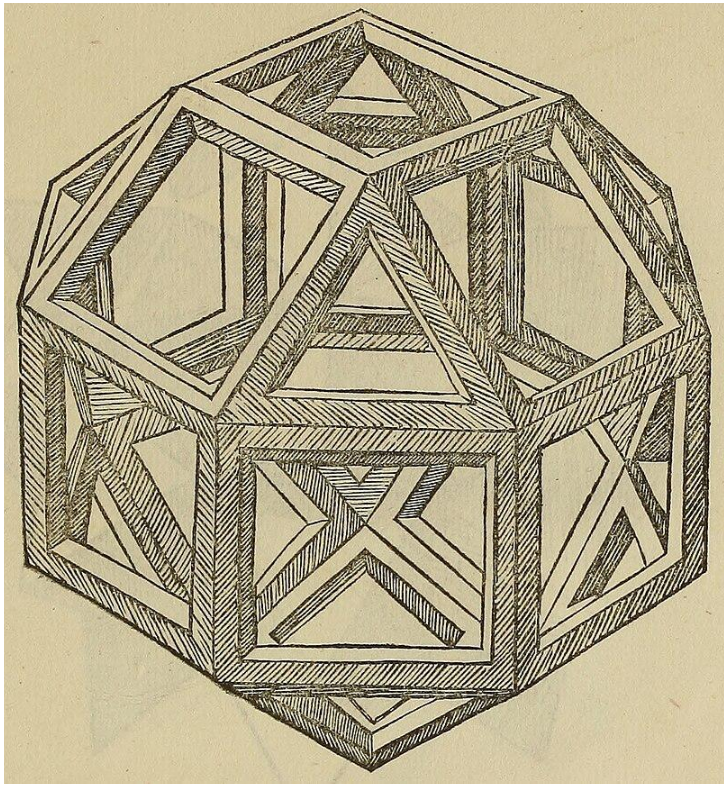


Michelangelo's plan

Caroline Karpinski (2010). Archimedes Salutes Bramante in a Draft for the School of Athens. Artibus et Historiae, Cracow, Iss. 61, 115-132.

The "Raphael Ambrosiana" or Raphael's cartoon for "The School of Athens", 1509, the Pinacoteca Ambrosiana in Milan.

Architecturally suggestive polyhedra



Rhombicuboctahedron

TEOB

SUMMARY AND CONCLUSIONS

- In the context of our previous research on geometrical forms of balance weights dated to the Early Islamic Period, we found one dirham weight with a complicated polyhedral shape in the Hecht Museum at the University of Haifa (Israel). We attributed it to a Truncated Elongated Octagonal Bipyramid (TEOB).

- Contrary to all Platonic solids and some Archimedean polyhedra, one cannot find any image or description of this polyhedron in the literature written by medieval Islamic or early Renaissance European mathematicians. The earliest Renaissance book containing the TEOB image is "La Pratica di Prospettiva", published in 1596 by Florentine architect and perspective artist Lorenzo Sirigatti.

- We presented modern examples of TEOB-like lamps and lanterns.

- Serendipitously, we revealed a drawing by Raphael with a lantern of a similar form.

ACKNOWLEDGMENTS

- **Dr. Vera Viana for bringing to my attention the image of TEOB in Lorenzo Sirigatti’s « La pratica di prospettiva »;**

- **Ms. Perry Harel (the Hecht Museum at the University of Haifa);**

- **Prof. Marco Milanesio (University of Eastern Piedmont) and Carlo Negroni Public Library’s employees in Novara;**
-
- **Prof. Edoardo Piccoli (Turin Polytechnic University), and Dr. Lyubava Chistova (State Hermitage Museum, St. Petersburg) for virtual discussions.**