On root systems, Carter diagrams and linkage systems

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Secret unity, not yet fully uncovered

- **Ian Stewart**: “It seemed completely mad. It seemed so mad, in fact, that Killing was rather upset that the exceptional groups existed, and for a time he hoped they were a mistake that he could eradicate. They spoiled the elegance of his classification. But they were ***there***, and we are finally beginning to understand ***why*** they are there. In many ways, the ***five exceptional Lie groups*** now look much more interesting than the four infinite families. They seem to be important in particle physics, as we will see; they are definitely important in mathematics. And they "have a secret unity, not yet fully uncovered...

- Exceptional Lie groups: E_6, E_7, E_8, F_4, G_2

"Why beauty is truth: a history of symmetry", **2007**

E_8 root system

- The **E_8 root system** is a rank 8 root system containing 240 root vectors spanning $\boldsymbol{R}^8$.

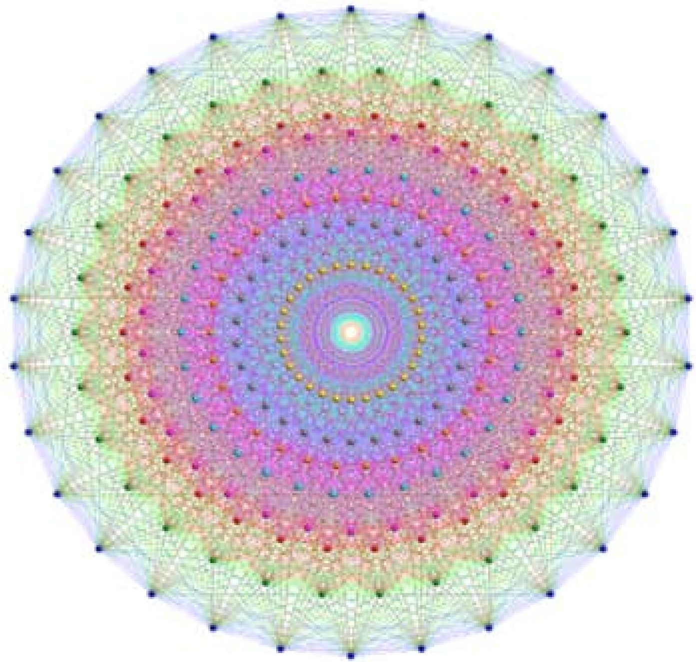
- Let us connect all non-orthogonal roots with each other. The graph thus obtained is the beautiful color computer-generated picture given on ***John Stembridge***'s home page (based on ***Peter McMullen's drawings* (1960)**).

- In the picture of the root system E_8, there are **6720 edges**.

https://websites.umich.edu/~jrs/coxplane.html

Stembridge – McMullen drawing

This picture is projection of the root system E_8 into the so-called Coxeter plane



Atlas of Lie Groups and Representations

- The ***Atlas of Lie Groups and Representations*** is a project to make available information about representations of reductive Lie groups. In particular: classifying all of the irreducible unitary representations of a given Lie group.
- ***David Vogan***: At 9 a.m. on January 8, 2007, a computer finished writing sixty gigabytes of files: Kazhdan-Lusztig polynomials for the split real group G(R) of type E_8. Their values at 1 are coefficients in irreducible characters of G(R). The biggest coefficient was 11,808,808,
- The Weyl group of E_8, which is the group of symmetries of the maximal torus (and root system) in the whole Lie group, has order $2^{14}\, 3^5\, 5^2\, 7$ = **696 729 600**.

Root system (definition)

A ***root system* Φ** in finite-dimensional vector space ***E*** is a finite set of non-zero vectors (called ***roots***) that satisfy the following conditions:

- The roots ***span E***.
- If a root $\alpha \in \Phi$ then $k\alpha \in \Phi$ only for $k = \pm 1$.
- For each $\alpha \in \Phi$, the set Φ is closed under ***reflection*** through the ***hyperplane perpendicular*** to α. perpendicular to α
- If $\alpha, \beta \in \Phi$, then the projection of β onto the line through α is an ***integer*** *or* ***half-integer*** multiple of α.

Reflections and Weyl group

- Reflection s_α :

$$s_\alpha(\beta) := \beta - 2(\alpha, \beta)/(\alpha, \alpha)$$

- Reflection s_α for simply-laced case:

$$s_\alpha(\beta) := \beta - (\alpha, \beta)$$

- The group generated by reflections through hyperplanes associated to the roots of Φ is called the ***Weyl group*** of Φ.

- Sizes of Weyl groups E_6 , E_7, E_8 :

E_6 – 51,840, E_7 – 2,903,040, E_8 – 696,729,600

Coxeter-Dynkin diagrams

- **H. S. M. Coxeter**: "The use of trees as diagrams for groups was anticipated in 1904, when C. Rodential was commenting on a set of models of cubic surfaces. He was analyzing the various rational double points that can occur on such a surface. In 1931 I used these diagrams in my ***enumeration of kaleidoscopes***, where the dots represent mirrors. E.B.Dynkin re-invented the diagrams in 1946 for the ***classification of simple Lie algebras***."

- Enumeration of kaleidoscopes is a classification system developed by H.S.M. Coxeter for finite ***reflection groups***. A standard kaleidoscope consists of multiple ***mirrors*** arranged in a chamber.

"The evolution of Coxeter-Dynkin diagrams", 1991

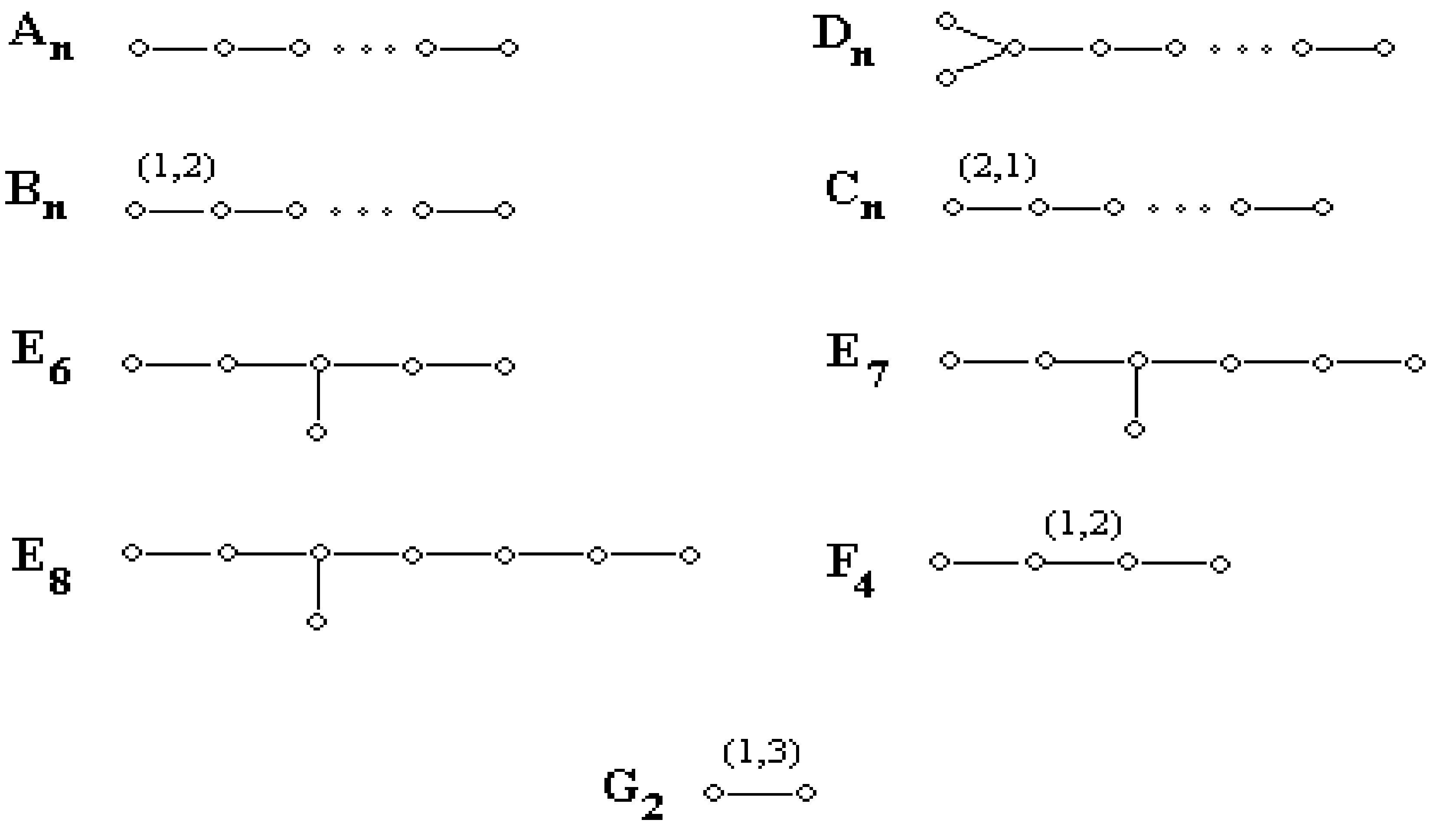
Using simple roots for semisimple Lie lagebras

- **E.B.Dynkin:** "I worked at Gelfabd seminar on Lie groups. Gelfand requested that I review the H. Weyl - Van der Waerden papers on semisimple Lie groups. I found them very difficult to read, and I tried to find my own ways. It came to my mind that there is a natural way to select a set of generators for a semisimple Lie algebra by using ***simple roots*** (i.e., ***roots which cannot be represented as a sum of two positive roots***). Since the angle between any two simple roots can be equal only to /2, 2/3, 3/4, 5/6, a system of simple roots can be represented by a simple diagram. An article was submitted to *Mat. Sbornik* in October 1944. Only a few years later, when recent literature from the West reached Moscow, I discovered that similar diagrams have been used by Coxeter for ***describing crystallographic groups***."

- A set of simple roots is a ***basis*** for the root system. Any root can be expressed as a ***sum of simple roots***.

"Foreword in "Selected papers of E. B. Dynkin", 2000

Simple roots as vertices of Dynkin diagrams



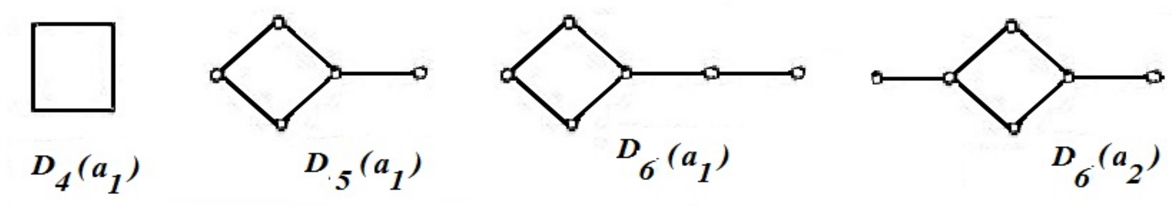
Admissible diagrams introduced by R.Carter

- In 1972, R. Carter introduced the so-called ***admissible diagrams*** representing elements of conjugacy classes in the Weyl group.

- A diagram Γ is said to be ***admissible*** if

  (a) The nodes of G *correspond to a set of linearly independent roots in* Φ

  (b) If a subdiagram of G is a cycle, *then it contains an* ***even number of nodes***



"Conjugacy classes in the Weyl group ", 1972

Surprising cycles, dotted and solid edges

- The ***presence of cycles*** was ***unexpected***, since the extended Dynkin diagram $\tilde{A}_n$ cannot be part of any admissible diagram due to the fact that we are within a finite root system.

- It turned out that the cycles in admissible diagrams <u>essentially differ</u> from the cycle $\tilde{A}_n$. Namely, in such a cycle, there must be two pairs of roots: A pair with a ***positive inner product*** together with a pair with a ***negative inner product***.

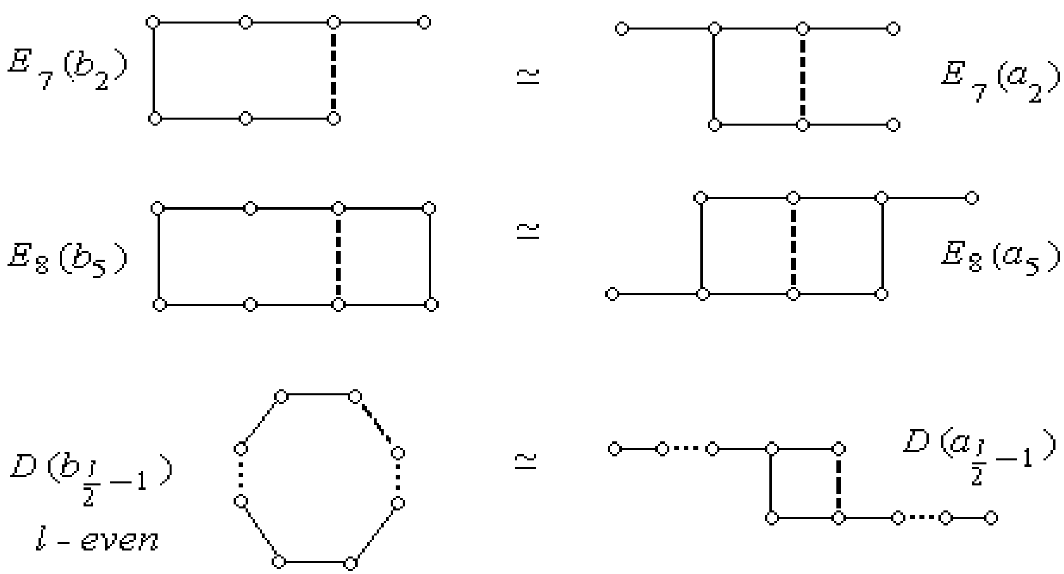
- This observation prompted me to distinguish such pairs of roots: draw ***dotted*** *(resp.* ***solid****) edge* $\{\alpha, \beta\}$ *if* $(\alpha, \beta) > 0$ *(resp.* $(\alpha, \beta) < 0$*)*.

- The admissible diagrams with dotted and solid edges are called ***Carter diagrams.*** The Carter diagram is a generalization of Dynkin diagrams that allows cycles of even length.

Equivalence of Carter Diagrams: Exclusion of long cycles

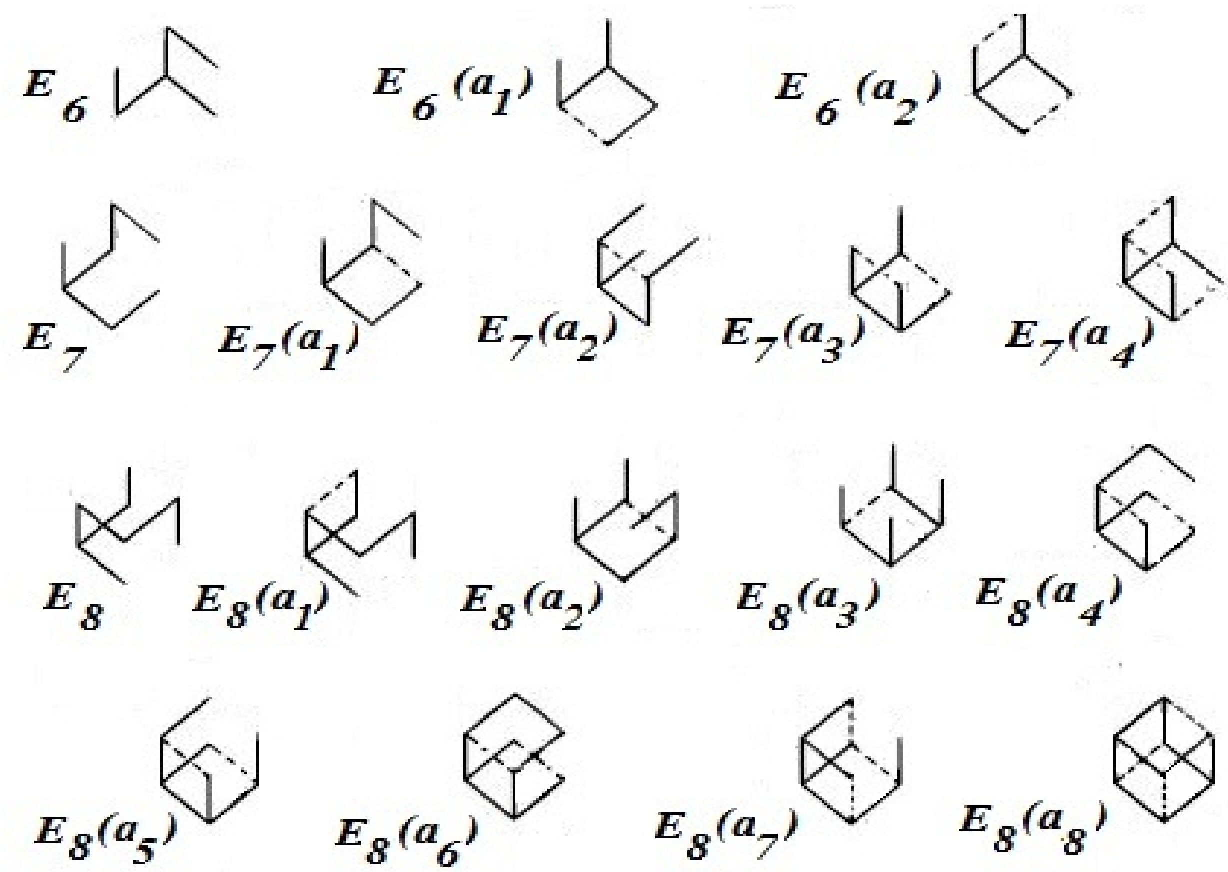
Lemma 1. *Every cycle in the Carter diagram contains* ***at least one solid*** *edge and* ***at least one dotted*** *edge.*

Theorem 1. *Any Carter diagram containing l-cycles, where $l > 4$, is* ***equivalent*** *to another Carter diagram containing* ***only 4-cycles***.



***"Equivalence of Carter Diagrams"*, 2017**

Homogeneous classes of Carter diagrams E_6, E_7, E_8



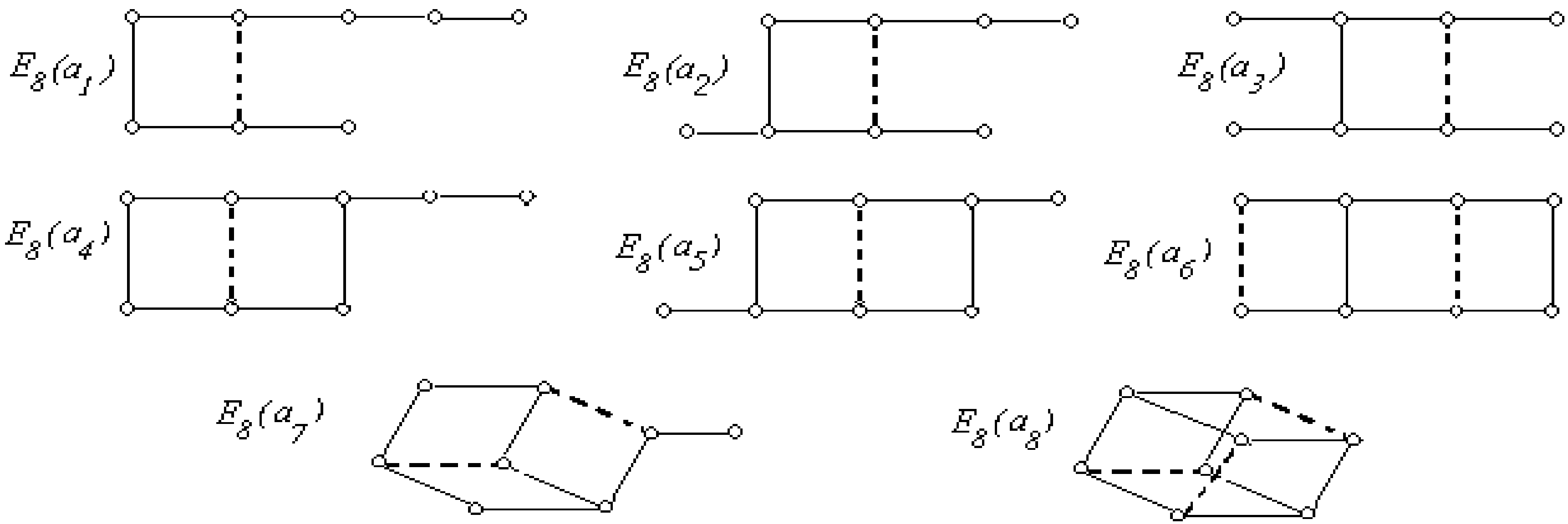
McKee-Smyth diagrams: Eigenvalues in (−2, 2)

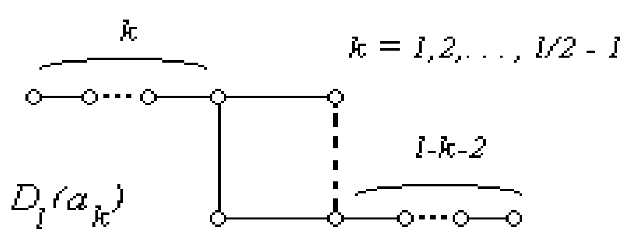
- Complete list of 8-vertex Carter diagrams with circles can be found in the paper of McKee and Smyth.
- ***Signed*** graphs. The {-1,0,1} -matrix with zeros on the diagonal is regarded as adjacency matrix of graph,where a non-zero (α, β)th entry denotes -1 or 1 on the edge corresponding vertices α and β . The signed graphs exactly correspond to diagrams with ***solid*** and ***dotted*** edges.
- McKee-Smyth's theorem: the signed graphs maximal with respect to having all their eigenvalues in (−2, 2) coincide

Exactly with Carter diagrams $E_8(a_i)$, $1 \leq i \leq 8$ and $D_l(a_i)$, $i < l/2 - 1$.

"Integer symmetric matrices having all their eigenvalues in the interval **[−2, 2]", 2007**

Carter diagrams $E_8(a_i)$, $1 \leq i \leq 8$





Linkage root, linkage diagram and linkage label vector

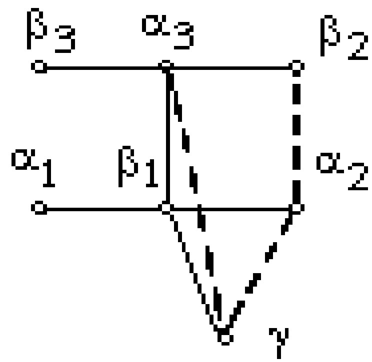
- The Carter diagram Γ is called ***bicolored*** if Γ-set of roots S can be partitioned into two disjoint subsets $S_\alpha = \{\alpha_1 \cdots \alpha_k\}$ and $S_\beta = \{\beta_1 \cdots \beta_h\}$, where roots of S_α (resp. S_β) are mutually orthogonal. The partition $S = S_\alpha \cup S_\beta$ is called ***bicolored partition*****.**

- Consider the extension of the root subset S by means of the root $\gamma \in \Phi$, so that the set of roots $S' = S \cup \gamma$ is ***linearly independent*** **.**

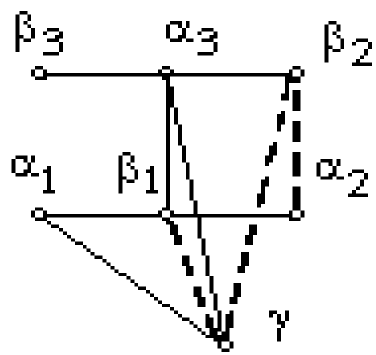
- The new diagram Γ' is obtained by addition ***solid edges*** if $(\gamma, \tau_i) = -1$ and ***dotted edges*** if $(\gamma, \tau_i) = 1$, where $\tau_i \in S'$. The diagram Γ' is said to be a ***linkage diagram*****.**

- The vector $\{(\gamma, \tau_i) \mid \tau_i \in S\}$ is called the ***linkage label vector*** and is denoted by $\boldsymbol{\gamma}^{\nabla}$.

Examples: linkage diagrams and linkage label vectors for $E_6(a_1)$



$$\gamma^{\nabla} = \begin{array}{c} \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{array}$$



$$\gamma^{\nabla} = \begin{array}{c} \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{array}$$

Linkage root vector

- Let Γ be a Carter diagram, and S be some Γ-set. What root γ can be added to S so that $\{S \cup \gamma\}$ is a set of ***linearly independent roots***? Such a root is called a ***linkage root vector***.

- The quadratic form $\mathbf{B}^{\vee}$ corresponding to the matrix $\mathbf{B}^{-1}$ is called the ***inverse quadratic form***. Both, the quadratic form $\mathbf{B}^{\vee}$ and matrix $\mathbf{B}^{-1}$ depend on the diagram Γ, which, for simplicity, is omitted.

Theorem 2. Let γ^{∇} be the linkage label vector corresponding to a root γ. The root γ is a linkage root ***if and only if***

$$\mathbf{B}^{\vee}(\gamma^{\nabla}) < 2.$$

The vertex extension of a Carter diagram

- Let Γ be one of simply-laced Dynkin diagrams, Γ' be one of the Carter diagram out of the homogeneous class $C(\Gamma)$.
Consider any Γ-set S and Γ'-set S' obtained by the transformation constructed by the ***transition theorem*** (see reference below), then $\text{Span}(S) = \text{Span}(S')$.

- Let $\hat{G}$ be a Dynkin diagram with the root system $\Phi(\hat{G})$ such that
  (1) $\text{rank}(\Phi(\hat{G})) = \text{rank}(\Phi(\Gamma)) + 1$
  (2) $\text{Span}(S) \subset \text{Span}(\Phi(\hat{G}))$

The Dynkin diagram $\hat{G}$ is said to be the ***vertex extension*** of Γ'.
The choose of the root system $\hat{G}$ is ambiguous.

"Transitions between root subsets associated with Carter diagrams.", 2022

Weyl group of quadratic form $\mathbf{B}^{\vee}$

- The ***Weyl group*** $W^{\vee}$ of $\mathbf{B}^{\vee}$ is generated by dual reflections s_i^*:
  $s_i^* \gamma^{\nabla} := \gamma^{\nabla} - (\gamma^{\nabla}, \tau_i)\tau_i^{\nabla}$

- Values $(s_i^* \gamma^{\nabla})_k$ belong to $\{-1, 0, 1\}$.

- The action of reflections s_i and dual reflections s_i^* related as follows: $(s_i \gamma)^{\nabla} = s_i^* \gamma^{\nabla}$.

- Reflections s_i^* ***preserve the quadratic form*** $\mathbf{B}^{\vee}$

. $\mathbf{B}^{\vee}(s_i^* \gamma^{\nabla}) = \mathbf{B}^{\vee}(\gamma^{\nabla})$ for linkage label vector γ^{∇}

Partial and full linkage systems

- Denote by $\mathscr{L}_{\hat{G}}(\Gamma', S')$ the set of linkage diagrams as follows:

  $$\mathscr{L}_{\hat{G}}(\Gamma', S') = \{ \gamma^{\nabla} \mid \gamma \in \Phi(\hat{G}), \gamma \notin \mathrm{Span}(S') \},$$

where $\Phi(\hat{G})$ is the root system associated with $\hat{G}$, and S' is some Γ'-set.

Proposition 1. The set $\mathscr{L}_{\hat{G}}(\Gamma', S')$ does not depend on choosing Γ'-set S'.

- The set $\mathscr{L}_{\hat{G}}(\Gamma', S')$ will be denoted by $\mathscr{L}_{\hat{G}}(\Gamma')$, and is called

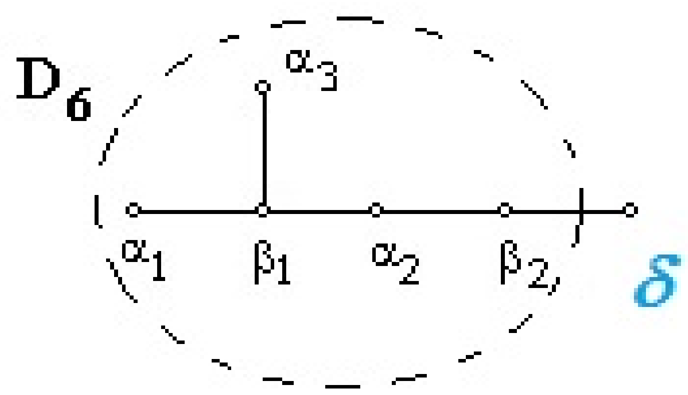
partial linkage system. The union of all partial systems by all possible vertex extensions is said to be the ***full linkage system*** *or* ***linkage system*** *of* Γ'.

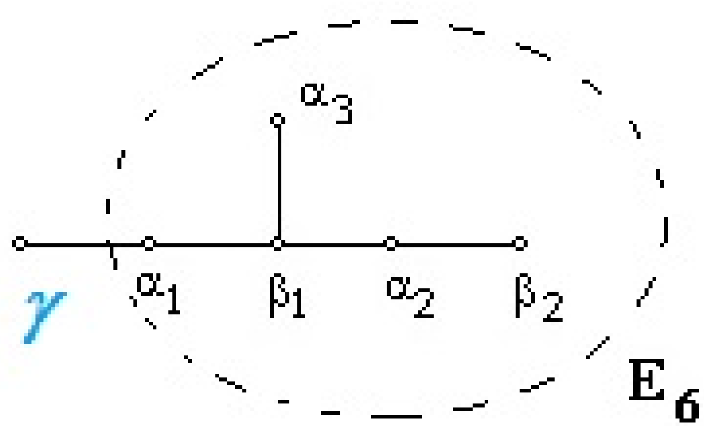
$$\mathscr{L}(\Gamma') = \bigcup \mathscr{L}_{\hat{G}}(\Gamma').$$

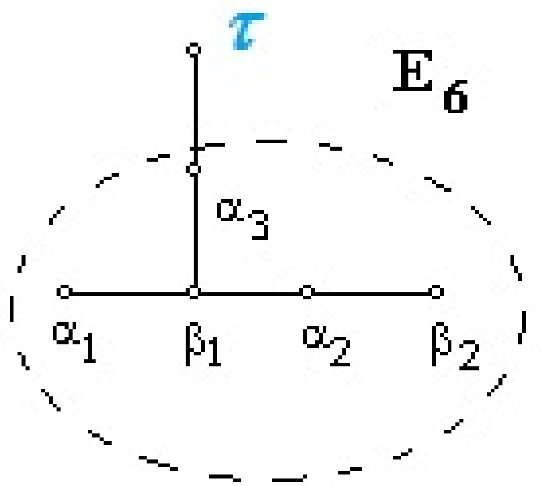
Proposition 2. For the homogeneous Carter diagrams Γ and Γ' the sizes of the full linkage systems are the same:

$$|\mathscr{L}(\Gamma')| = |\mathscr{L}(\Gamma)|$$

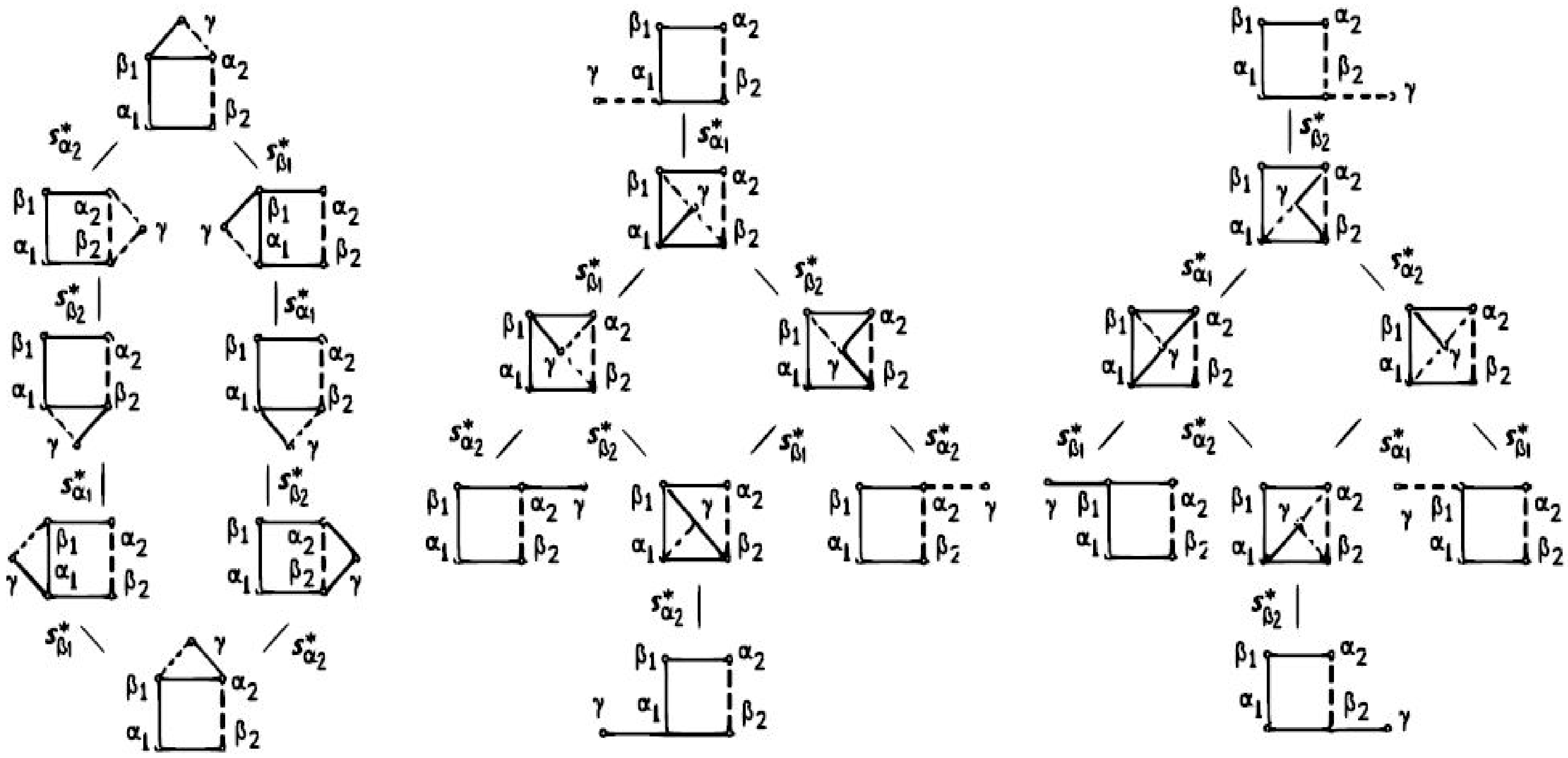
Example: Three vertex extensions of D5



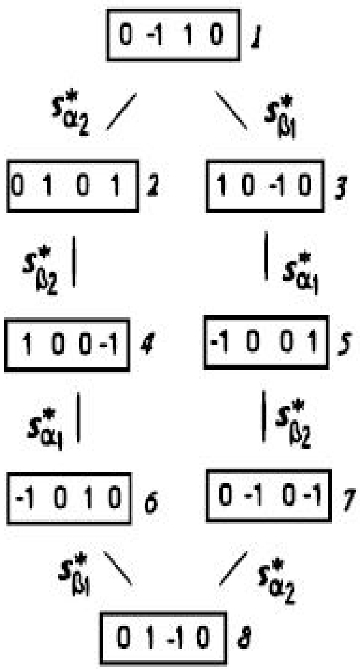


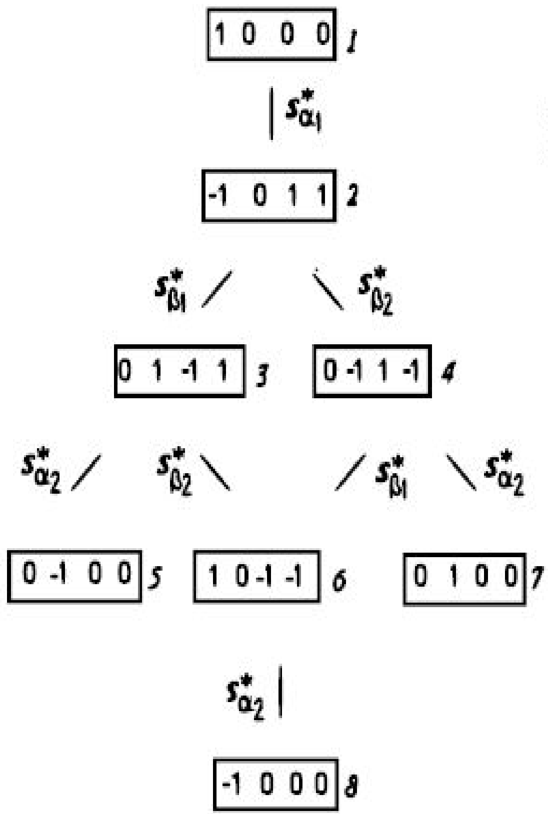


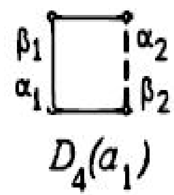
Three components linkage diagrams of $\mathscr{L}(D_4(a_1))$

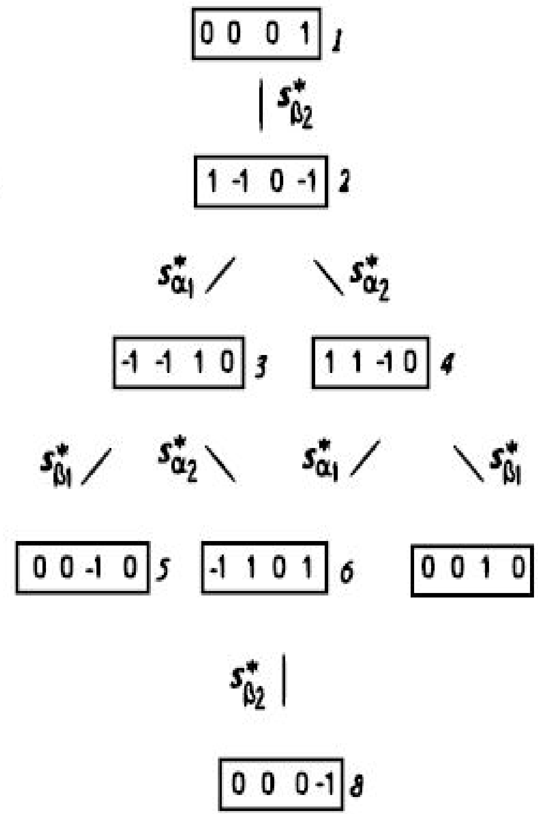


Three components linkage label vectors of $\mathscr{L}(D_4(a_1))$

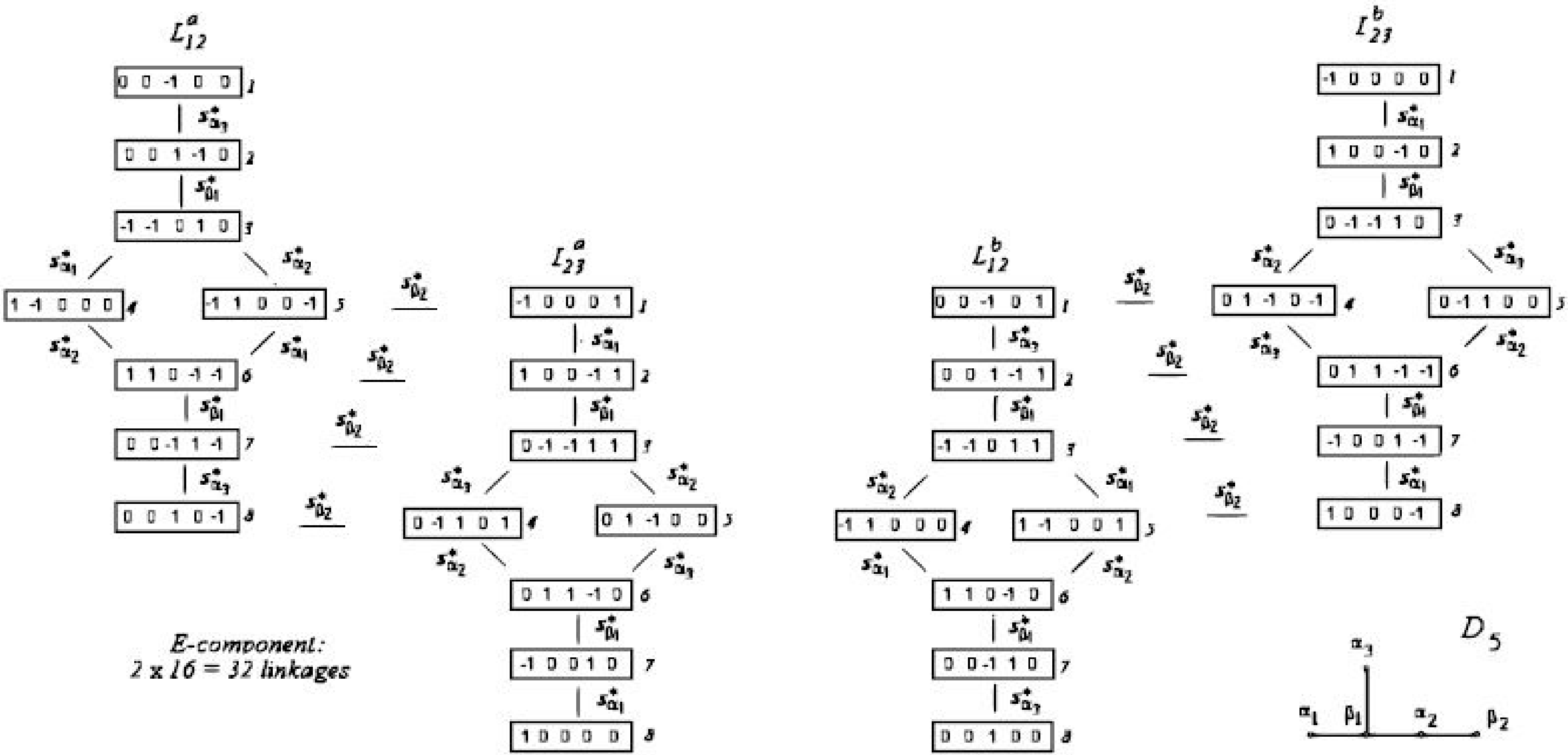






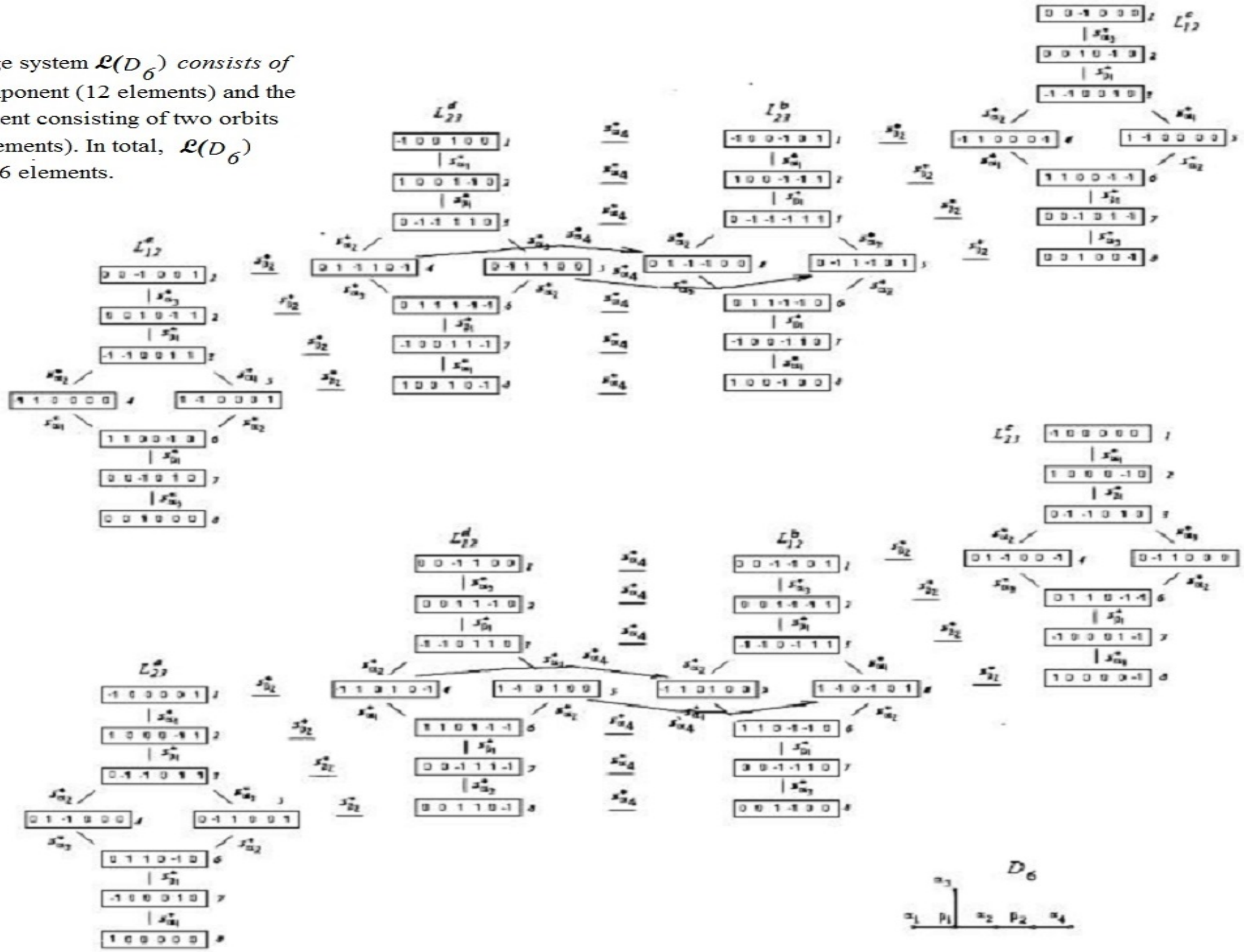


Two components linkage label vectors of $\mathbb{L}(D_5)$



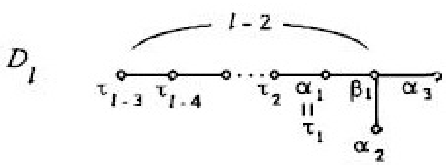
Two E-components of linkage label vectors of $\mathbf{L}(D_6)$

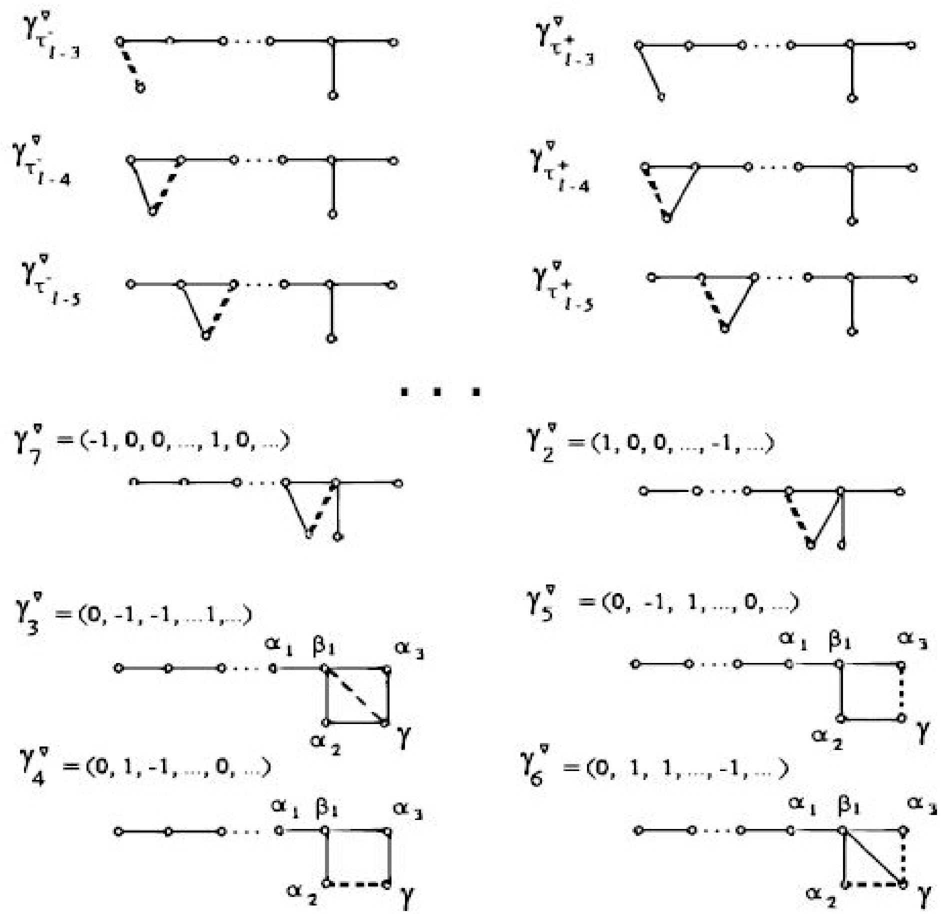
The linkage system $\mathcal{L}(D_6)$ *consists of* the D-component (12 elements) and the E-component consisting of two orbits (2 x 32 elements). In total, $\mathcal{L}(D_6)$ contains 76 elements.



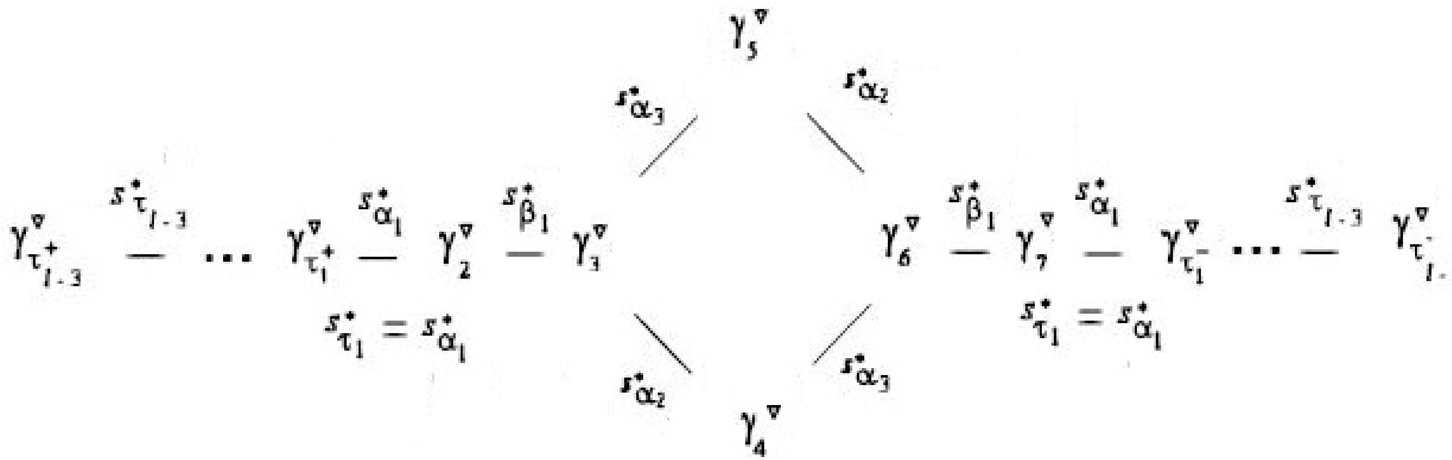
The D-component of linkage diagrams of $\mathcal{L}(D_l)$

The linkage system $\mathcal{L}(D_l)$ has the single D-component that contains $2l$ linkage diagrams.





The D-component of linkage system of $\mathscr{L}(D_l)$



The linkage system $\mathscr{L}(D_l)$ has spindle-like shape that contains $2l$ linkage diagrams

$\gamma^\nabla_2 = (1, 0, 0, \dots, -1, \dots)$ $\quad$ $\gamma^\nabla_6 = (0, 1, 1, \dots, -1, \dots)$

$\gamma^\nabla_3 = (0, -1, -1, \dots, 1, \dots)$ $\quad$ $\gamma^\nabla_7 = (-1, 0, 0, \dots, 1, 0, \dots)$

Number of linkage diagrams for D_l and $D_l(a_k)$

Γ	Number of components	Number of linkage diagrams, and $p = \mathscr{B}^{\vee}_{\Gamma}(\gamma^{\nabla})$		
		D-components	E-components	In all
		$p = 1$	$p = \frac{l}{4}$	
D_4, $D_4(a_1)$	3	$3 \times 8 = 24$	-	24
D_5, $D_5(a_1)$	3	10	$2 \times 16 = 32$	42
D_6, $D_6(a_1)$, $D_6(a_2)$	3	12	$2 \times 32 = 64$	76
D_7, $D_7(a_1)$, $D_7(a_2)$	3	14	$2 \times 64 = 128$	142
D_l, $D_l(a_k)$, $l > 7$	1	$2l$	-	$2l$

Enhanced Dynkin diagrams introduced by Dynkin-Minchenko

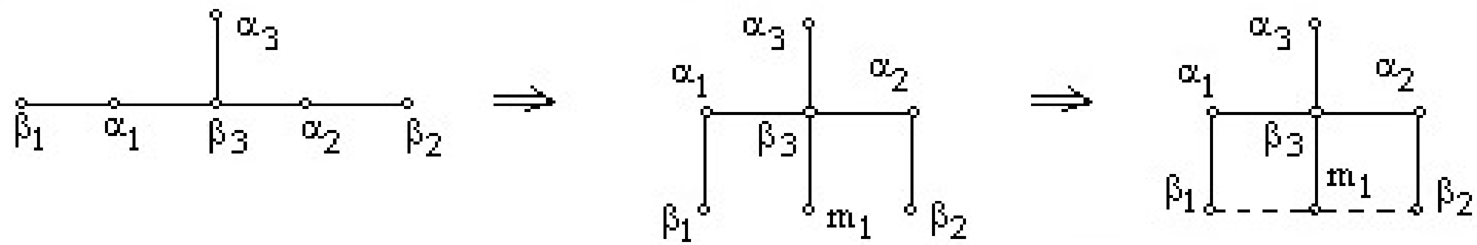
- Let Γ be a Dynkin diagram of the complex semisimple Lie algebra $\mathfrak{g}$. Subdiagrams of Γ are Dynkin diagrams of regular subalgebras of $\mathfrak{g}$. However, not all regular subalgebras can be obtained in this way.

- These problems are solved by Dynkin and Minchenko using so-called ***enhanced*** Dynkin diagrams. They constructed an enhancement of Γ by a recursive procedure which they call the ***completion***.

- At each step of the procedure, find a $\boldsymbol{D_4}$-subset in the already introduced nodes, add the minimal root of this subset, and connect it by edges to the corresponding part of the already introduced nodes.

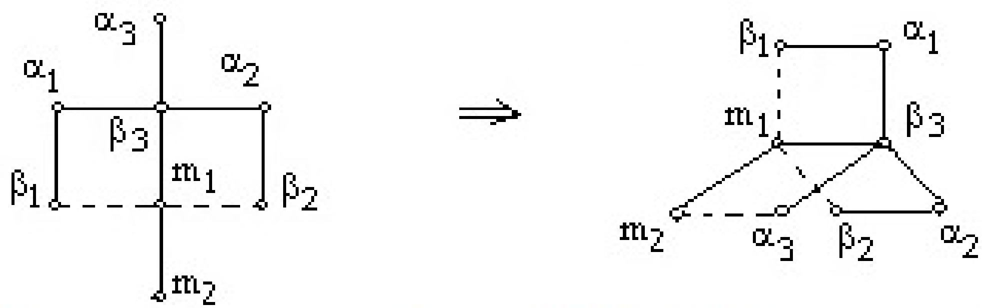
"Enhanced Dynkin diagrams and Weyl orbits", 2010.

Example: completion procedure for E_6



Add $m_1 = -(\alpha_1 + \alpha_2 + \alpha_3 + 2\beta_3)$ *and connect* m_1 *to* β_1 *and* β_2.

It is easy to check that $(m_1, \alpha_i) = 0$ *for* $i = 1,2,3$ *and* $(m_1, \beta_i) = 1$ *for* $i = 1,2$.



Further, add $m_2 = -(-\beta_1 - \beta_2 + \beta_3 + 2m_1)$. *We have* $(m_2, \beta_i) = 0$ *for* $i = 1,2,3$ *and* $(m_2, \alpha_i) = 0$ *for* $i = 1,2$. *Connect* m_2 *to* α_3. *Completion done.*

Signed enhanced Dynkin diagrams

- Vavilov and Migrin merged the enhanced Dynkin diagrams approach with the solid/dotted edge concept of Carter diagrams. They called these diagrams ***signed enhanced Dynkin diagrams***.

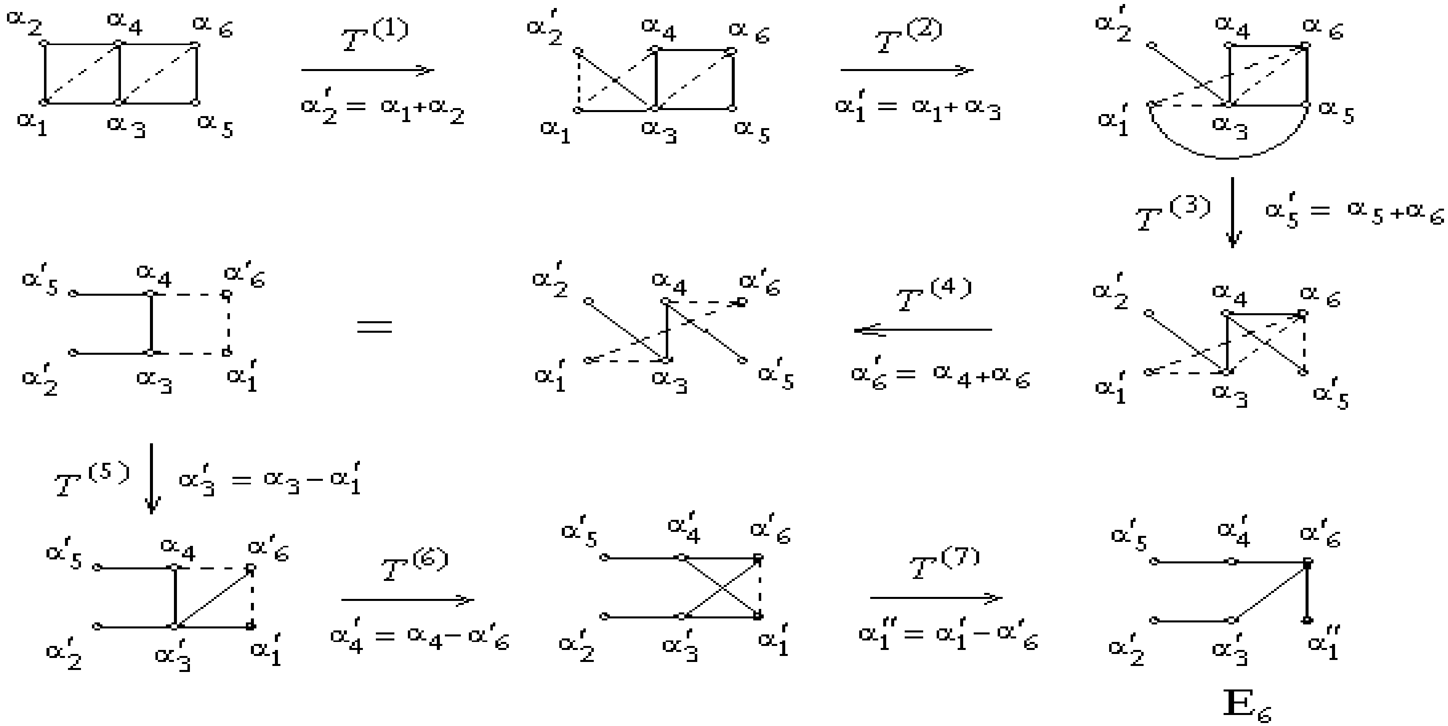
- **Theorem** (**Vavilov-Migrin**). The signed enhanced Dynkin diagrams of types E_6, E_7, E_8 contain all Carter diagrams of conjugacy classes of the Weyl groups $W(E_6)$, $W(E_7)$, $W(E_8)$.

- They provide an a posteriori observation of this fact.

***"Enhanced Dynkin diagrams done right"*, 2021**

Gabrielov's example corresponding to a singularity $x^4 + y^3 + z^2$.

Changing the basis corresponding to a singularity. He studied the quadratic forms associated with singularities and used technique of solid/dotted edges with the same sense.



"Intersection matrices for certain singularities", 1973

Integral quadratic form and solid/dotted edges

♦ The presence of a dotted or solid edge reflects the fact that the ***integral quadratic form*** (Tits form) takes values +1 or -1 at the roots connected by the given edge.

♦ Another contexts for integral quadratic form and solid/dotted edges: de Graaf and Elsahvili used the Carter diagrams with solid and dotted edges for description of ***nilpotent elements*** in semisimple Lie algebras.

"Induced Nilpotent Orbits of the Simple Lie Algebras of Exceptional Type", 2009

"Integer quadratic forms and extensions of subsets of linearly independent roots" 2025

Thank you!